

# The Asymptotic Power and Relative Efficiency of Some c-Sample Rank Tests of Homogeneity Against Umbrella Alternatives

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## Abstract

For the c-sample location problem with umbrella alternatives we compare some generalizations of the test of Mack and Wolfe (1981). All the tests are based on pairwise ranking methods. The asymptotic power and asymptotic relative efficiency of the so-called Mack-Wolfe-type test, Tryon-Hettmansperger-type test and Puri-type test are derived and compared with modifications of the test of Hettmansperger and Norton (1987) where the ranks are taken over all samples.

The Mack-Wolfe-type test and the Tryon-Hettmansperger-type test are further generalized by introducing weight coefficients for the substatistics. For the case of a specified alternative these weights are determined in such a way that the efficacies become maximal. It is shown that the maximal achievable efficacies in the defined classes of the generalized Mack-Wolfe-type test and Tryon-Hettmansperger-type test always are equal.

The case of an unknown peak is briefly discussed.

*Keywords: Mack-Wolfe-type test; Tryon-Hettmansperger-type test; Hettmansperger-Norton-type test; Chen-Wolfe-type test; Efficacy.*

## 1. INTRODUCTION

Let  $X_{i1}, \dots, X_{in_i}$ ,  $i = 1, \dots, c$ , be independent random samples from a population with an absolutely continuous distribution function  $F(x - \vartheta_i)$ ,  $\vartheta_i \in \mathbf{R}$ . In the following we assume that  $F$  is twice continuously differentiable on  $(-\infty, \infty)$  except for a set of Lebesgue measure zero;  $f'$  denotes

the derivative of the density  $f$  where it exists and it is defined to be zero, otherwise. The Fisher information  $I(f)$  is assumed to exist. We wish to test:

$$H_0 : \vartheta_1 = \dots = \vartheta_c$$

against one of the following umbrella alternatives

$$H_{1A} : \vartheta_1 \leq \dots \leq \vartheta_l \geq \vartheta_{l+1} \geq \dots \geq \vartheta_c$$

with at least one strict inequality and given  $l$  or

$$H_{1B} : \vartheta_1 \leq \dots \leq \vartheta_l \geq \vartheta_{l+1} \geq \dots \geq \vartheta_c$$

with at least one strict inequality and given  $l$  and specified  $\tilde{\vartheta} = (\vartheta_1, \dots, \vartheta_c)$

or

$$H_{1C} : \bigcup_{l=1}^c \{(\vartheta_1, \dots, \vartheta_c) : \vartheta_1 \leq \dots \leq \vartheta_l \geq \vartheta_{l+1} \dots \geq \vartheta_c\}$$

with at least one strict inequality (the case of unknown peak).

Obviously,  $H_{1B} \subset H_{1A} \subset H_{1C}$ . At first we consider the problem  $(H_0, H_{1A})$  where the peak  $l$  is assumed to be known. Furthermore we assume  $l \neq 1$  and  $l \neq c$ . If  $l = 1$  or  $l = c$  the problem reduces to the case of ordered alternatives, see Büning and Kössler (1999) and Kössler and Büning (1996). However, the results of section 2 can easily be extended to the cases  $l = 1$  or  $l = c$ .

The nonparametric test of Mack and Wolfe (1981) may be the most familiar test for such umbrella alternatives. In section 2 modifications of the tests of Mack and Wolfe (1981), Puri (1965), Tryon and Hettmansperger (1973) and Hettmansperger and Norton (1987) are considered. The so-called Mack-Wolfe-type tests have been introduced by Büning and Kössler (1997) where it is shown that these tests have good small and moderate sample size properties. Local alternatives of the form  $\vartheta_{iN} = \Delta\theta_i/\sqrt{N}, i = 1, \dots, c$  are considered and the asymptotic efficacies of the Mack-Wolfe-type test (MWT-test), Puri-type test (PT-test) and the Tryon-Hettmansperger-type test (THT-test) are derived and compared with that of the Hettmansperger-Norton-type test (HNT-test). The efficacies of these tests depend on the underlying density  $f$  and the score generating function  $\phi$  as well as on the sample sizes and the special kind of alternatives. For an investigation of the asymptotic relative Pitman efficiencies (ARE) of the different types of rank tests the score generating function is assumed to be fixed. Only the effect of the sample sizes and the ARE is studied.

In section 3 the test statistics are further generalized by introducing weight coefficients for the substatistics occurring in the statistics  $MWT$  and  $THT$  given below. For the test problem  $(H_0, H_{1B})$  the weights can be determined in such a way that the efficacies become maximal. It is shown that the generalized versions of the tests MWT and THT, each of them based on the "optimal" weights, always have the same efficacies and therefore they are asymptotically equivalent in the sense of ARE.

In section 4 the test problem  $(H_0, H_{1C})$ , i.e. the case of an unknown peak, is briefly discussed and the asymptotic power of the Chen-Wolfe-type test is obtained analytically if  $c = 3$  and  $n_1 = n_3$ .

## 2. SOME TYPES OF RANK TESTS AND THEIR ASYMPTOTIC POWER AND EFFICACY

### 2.1 The Test Statistics

We consider the so called Mack-Wolfe-type test (Büning and Kössler, 1997), Tryon-Hettmansperger-type test, Puri-type test and Hettmansperger-Norton-type test. The first three tests are generalisations of the Mack-Wolfe test. The Mack-Wolfe test statistic is given by

$$MW = \sum_{r=1}^{l-1} \sum_{s=r+1}^l V_{rs} + \sum_{r=l}^{c-1} \sum_{s=r+1}^c V_{sr},$$

where  $V_{rs}$  is the usual two-sample Wilcoxon test statistic and  $V_{sr} = n_r n_s - V_{rs}$ . Under  $H_0$ , MW is asymptotically normal (see Randles and Wolfe, 1979, p. 397).

The statistic of the Mack-Wolfe-type test is defined by

$$MWT = \sum_{i=2}^l S_{(1\dots i-1)i} + \sum_{i=l}^{c-1} S_{(c\dots i+1)i}$$

with

$$S_{(1\dots i-1)i} = N_i \cdot \sum_{k=N_{i-1}+1}^{N_i} a_{N_i}(R_{1\dots i}^k), \quad i = 2, \dots, c$$

$$S_{(c\dots i+1)i} = (N - N_{i-1}) \cdot \sum_{k=N_{i-1}+1}^{N_i} a_{N-N_{i-1}}(R_{c\dots i}^k), \quad i = 1, \dots, c-1,$$

$a_{N_i}(k) \in \mathbf{R}$ ,  $N_i = \sum_{j=1}^i n_j$ ,  $1 \leq i \leq c$ ,  $N = N_c$ ,  $N_0 = 0$ .

$R_{1\dots i}^k$  is the rank of the  $k$ th observation in the *first* pooled  $i$  samples  $X_{11}, \dots, X_{1n_1}, \dots, X_{i1}, \dots, X_{in_i}$ .  $S_{(1\dots i-1)i}$  is a two-sample linear rank statistic computed on the  $i$ th sample versus the combined data in the first  $(i-1)$  samples.  $R_{c\dots i}^k$  is the rank of the  $k$ th observation in the *last* pooled  $(c-i+1)$  samples  $X_{i1}, \dots, X_{in_i}, \dots, X_{c1}, \dots, X_{cn_c}$ .  $S_{(c\dots i+1)i}$  is a two-sample linear rank statistic computed on the  $i$ th sample versus the combined data in the last  $(c-i)$  samples. The weights  $N_i$  and  $(N - N_{i-1})$  in the first and in the second sum respectively, are chosen to have a simpler formula for the asymptotic variance of  $MWT$  which will be given later on.

The Puri-type test and the Tryon-Hettmansperger-type test as modifications of the tests of Puri (1965) and Tryon and Hettmansperger (1973) are defined similarly to the case of ordered alternatives (cf. Büning and Kössler 1999).

The Puri-type test statistic is given by

$$PT = \sum_{1 \leq i < j \leq l} U_{i,j} + \sum_{l \leq j < i \leq c} U_{i,j}$$

with

$$U_{i,j} = (n_i + n_j) \sum_{k=n_i+1}^{n_i+n_j} a_{n_i+n_j}(R_{ij}^k),$$

$a_{n_i+n_j}(k) \in \mathbf{R}$ ,  $1 \leq i, j \leq c$  and  $R_{ij}^k$  is the rank of the  $k$ th observation in the combined two samples  $X_{i1}, \dots, X_{in_i}$  and  $X_{j1}, \dots, X_{jn_j}$ . Note that a similar statistic was proposed by Archambault, Mack and Wolfe (1977) but they use different weights for the  $U_{i,j}$ .

The Tryon-Hettmansperger-type test statistic is given by

$$THT = \sum_{i=1}^{l-1} U_{i,i+1} + \sum_{i=l}^{c-1} U_{i+1,i}$$

with  $U_{i,j}$  defined as above.

Notice, that  $PT$  is based on  $\binom{l}{2} + \binom{c-l+1}{2}$  two-sample comparisons whereas  $MWT$  and  $THT$  include only  $(l-1) + (c-l) = c-1$  such comparisons.

Hettmansperger and Norton (1987) propose a test for umbrella alternatives which is based on the ranks  $R_{ij}$  of  $X_{ij}$  in the pooled sample  $X_{11}, \dots, X_{cn_c}$ . Replacing the ranks  $R_{ij}$  by arbitrarily chosen scores  $a_N(R_{ij})$  we get the so-called Hettmansperger-Norton-type test the statistic of which is defined by

$$HNT = \sum_{i=1}^c v_i T_{i,N},$$

where

$$T_{i,N} = \sum_{j=1}^{n_i} a_N(R_{ij}),$$

$$v_i = \frac{1}{N} \cdot \begin{cases} i - \bar{v}_N & \text{if } i \leq l \\ 2l - i - \bar{v}_N & \text{else,} \end{cases} \quad \text{with} \quad \bar{v}_N = \frac{1}{N} \left[ \sum_{i=1}^l i n_i + \sum_{i=l+1}^c (2l - i) n_i \right]$$

and  $a_N(k) \in \mathbf{R}$ .

The (exact or asymptotic) associated  $\alpha$ -level tests reject  $H_0$  in favour of  $H_1$  ( $H_{1A}$  or  $H_{1B}$ ) if  $MWT, PT, THT$  or  $HNT$  are at least as large as the upper  $\alpha$ -quantile of the (exact or asymptotic) null distribution of  $MWT, PT, THT$  or  $HNT$ , respectively. For convenience the corresponding tests are called MWT-test, PT-test, THT-test and HNT-test.

It is assumed that the scores  $a_L$  ( $L = N_i$  and  $L = N - N_{i-1}$  for  $MWT$ ,  $L = n_i + n_j$  for  $PT$  and  $TT$ ,  $L = N$  for  $HNT$ ) in the definition of the test statistics above are generated by an absolutely continuous score function  $\phi$  with  $\lim_{L \rightarrow \infty} a_L(1 + [uL]) = \phi(u)$ ,  $0 < u < 1$ , where  $\phi$  is associated with a density function  $g$  given by

$$\phi(u, g) := \phi(u) = -\frac{g'(G^{-1}(u))}{g(G^{-1}(u))}. \quad (1)$$

The function  $\phi(u, g)$  is the so called optimal score function of the density function  $g$  with quantile function  $G^{-1}$ . It is assumed that

$$\bar{\phi} := \int_0^1 \phi(u, g) du = 0 \quad (2)$$

and the Fisher information  $I(g)$  is finite, i.e.

$$I(g) := \int_0^1 \phi^2(u, g) du < \infty. \quad (3)$$

For convenience we use the notations

$$d(f, g) := \int_0^1 \phi(u, g)\phi(u, f) du$$

and

$$C(f, g) := d(f, g) \cdot I(g)^{-1/2}$$

It is assumed throughout the paper that the score function  $\phi(u, g)$  and therefore the factors  $d(f, g)$  and  $C(f, g)$  are fixed. For the choice of the function  $\phi(u, g)$  and for an investigation of the associated factor  $C(f, g)$  we refer to Büning and Kössler (1999 and 1996).

## 2.2 The asymptotic power and efficacies of the test statistics

Under the assumptions (??) and (??) the limiting distributions of  $MWT/\sigma_M$ ,  $PT/\sigma_P$ ,  $THT/\sigma_T$  and  $HNT/\sigma_H$  are, under  $H_0$ , asymptotically standard normal where the asymptotic variances  $\sigma_M^2$ ,  $\sigma_P^2$ ,  $\sigma_T^2$  and  $\sigma_H^2$  of  $MWT$ ,  $PT$ ,  $THT$  and  $HNT$  are given by

$$\begin{aligned} \sigma_M^2 &= I(g) \cdot \left[ \frac{1}{3} \left( N_l^3 - \sum_{i=1}^l n_i^3 + (N - N_{l-1})^3 - \sum_{i=l}^c n_i^3 \right) + 2N_{l-1}n_l(N - N_l) \right] \\ &= \sigma_P^2 \\ \sigma_T^2 &= I(g) \cdot \left[ \sum_{i=1}^{c-2} n_i n_{i+1} (n_i + n_{i+1} - 2n_{i+2}) \right] + \\ &\quad + I(g) \cdot [n_{c-1}n_c(n_{c-1} + n_c) + 4n_{l-1}n_l n_{l+1}] \\ \sigma_H^2 &= \sum_{i=1}^c n_i v_i^2 I(g), \end{aligned}$$

cf. Büning and Kössler (1997) for  $MWT$ , Puri (1965) for  $PT$  and Hájek and Šidák (1967, theorem 5.1.6a) for  $HNT$ . The formula for  $\sigma_T^2$  can easily be obtained from Büning and Kössler (1999) by introducing the term  $2\text{cov}(U_{l-1,l}, U_{l+1,l})$ .

The asymptotic normality is obtained under the following assumptions **(A)**:

Let be  $\Delta > 0$  and  $\{\vartheta_{1N}, \dots, \vartheta_{cN}\}$  a sequence of "near" alternatives with  $N^{1/2}\vartheta_{iN}/\Delta = \theta_i$ ,  $\theta_1 \leq \dots \leq \theta_l \geq \theta_{l+1} \geq \dots \geq \theta_c$ , and at least one strict inequality. Denote  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_c)$ ,  $\mathbf{n} = (n_1, \dots, n_c)$  and assume without loss of generality  $\theta_1 = 0$ . Let  $\min(n_1, \dots, n_c) \rightarrow \infty$ ,  $n_i/N \rightarrow \lambda_i$ ,  $0 < \lambda_i < 1$ ,  $i = 1, \dots, c$ .

**THEOREM 1:** Under assumptions **(A)** the statistics  $(MWT - \mu_M)/\sigma_M$ ,  $(PT - \mu_P)/\sigma_P$ ,  $(THT - \mu_T)/\sigma_T$  and  $(HNT - \mu_H)/\sigma_H$  have a limiting standard normal distribution with

$$\begin{aligned} \mu_M &= \Delta \cdot N^{-1/2} \cdot d(f, g) \cdot \left[ \sum_{i=2}^l \theta_i n_i (N_{i-1} + N_i - N_l) + \sum_{i=l}^c \theta_i n_i (N - N_i + N_{l-1} - N_{i-1}) \right] \\ \mu_P &= \Delta \cdot N^{-1/2} \cdot d(f, g) \cdot \left[ \sum_{1 \leq i < j \leq l} (\theta_j - \theta_i) n_i n_j + \sum_{l \leq j < i \leq c} (\theta_j - \theta_i) n_i n_j \right] \\ &= \mu_M \\ \mu_T &= \Delta \cdot N^{-1/2} \cdot d(f, g) \cdot \left[ \sum_{i=1}^{l-1} (\theta_{i+1} - \theta_i) n_i n_{i+1} + \sum_{i=l}^{c-1} (\theta_i - \theta_{i+1}) n_i n_{i+1} \right] \\ \mu_H &= \Delta \cdot N^{-1/2} \cdot d(f, g) \cdot \sum_{i=1}^c n_i v_i \theta_i. \end{aligned} \tag{4}$$

*Proof:* 1. For convenience we first summarize some results from Hájek and Šidák (1967, chs. 5,6). Define

$$W_i^* := \sum_{j=1}^{n_i} \phi(F(X_{ij})), \quad i = 1, \dots, c,$$

where  $\phi$  is the score function (??). The statistics  $W_i^*$  are, under  $H_0$ , sums of  $n_i$  iid random variables and therefore asymptotically normally distributed with asymptotic expectations zero and variances  $\sigma_i^{*2} = n_i I(g)$ . Denote

$$\gamma_{i',j}^i = \begin{cases} 1 & \text{if } i = i' \\ 0 & \text{else,} \end{cases} \quad \bar{\gamma}_i = \frac{n_i}{N}, \quad \bar{a}_N = \frac{1}{N} \sum_{i'=1}^c \sum_{j=1}^{n_{i'}} a_N(R_{i'j}),$$

and rewrite  $T_{i,N}$  as

$$T_{i,N} = \sum_{i'=1}^c \sum_{j=1}^{n_{i'}} (\gamma_{i',j}^i - \bar{\gamma}_i) a_N(R_{i'j}) + n_i \bar{a}_N.$$

Let be

$$\begin{aligned} \tilde{T}_{i,N} &:= \sum_{i'=1}^c \sum_{j=1}^{n_{i'}} (\gamma_{i',j}^i - \bar{\gamma}_i) \phi(F(X_{i'j})) + n_i \bar{a}_N \\ &\sim \frac{N - n_i}{N} W_i^* - \frac{n_i}{N} \sum_{i'=1, i' \neq i}^c W_{i'}^* + n_i \bar{a}_N. \end{aligned} \quad (5)$$

Since the  $W_i^*$  are independent, under  $H_0$ , the variance of the  $\tilde{T}_{i,N}$  can easily be computed

$$\sigma_i^2 := \text{var}(\tilde{T}_{i,N}) \sim \frac{n_i(N - n_i)}{N} \cdot I(g). \quad (6)$$

From Hájek and Šidák (1967, see proof of theorem 5.1.5a) we have, under  $H_0$ ,  $(T_{i,N} - \tilde{T}_{i,N})/\sigma_i \xrightarrow{p} 0$ , i.e.  $T_{i,N}/\sigma_i$  is asymptotically standard normal.

The asymptotic normality of  $\tilde{T}_{i,N}$  and of  $T_{i,N}$ , under  $H_A$ , both with variances  $\sigma_i^2$ , follows from Hájek and Šidák (1967, see proof of theorem 6.2.4). The asymptotic expectations of  $T_{i,N}$  and  $\tilde{T}_{i,N}$  are given by

$$E(T_{i,N}) \sim E(\tilde{T}_{i,N}) \sim n_i(\vartheta_{i,N} - \bar{\vartheta}_N) \cdot d(f, g), \quad (7)$$

where  $\bar{\vartheta}_N = \frac{1}{N} \sum_{i=1}^c n_i \vartheta_{i,N}$ .

2. Since  $\sum_{i=1}^c n_i v_i = 0$  we obtain from (??)

$$HNT \sim \sum_{i=1}^c v_i W_i^*$$

and  $HNT$  is asymptotically normal with expectation  $\mu_H$  and variance  $\sigma_H^2$ .

3. From Koziol and Reid (1977) we have that the two-sample statistics  $U_{i,j}$  are asymptotically equivalent to a linear combination of  $T_{i,N}$  and  $T_{j,N}$ , more precisely  $U_{i,j} \sim n_i T_{j,N} - n_j T_{i,N}$ . From (??) we have

$$U_{i,j} \sim n_i W_j^* - n_j W_i^*.$$



4. For the  $THT$ -statistic we obtain ( $n_0 = n_{c+1} = 0$ )

$$\begin{aligned}
THT &\sim \sum_{i=1}^{l-1} (n_i T_{i+1,N} - n_{i+1} T_{i,N}) + \sum_{i=l}^{c-1} (n_{i+1} T_{i,N} - n_i T_{i+1,N}) \\
&= \sum_{i=1}^{l-1} (n_{i-1} - n_{i+1}) T_{i,N} + (n_{l-1} + n_l) T_{l,N} + \sum_{i=l+1}^{c-1} (n_{i+1} - n_{i-1}) T_{i,N} \\
&\sim \sum_{i=1}^c \alpha_{T,i} W_i^*
\end{aligned}$$

with

$$\alpha_{T,i} = \begin{cases} n_{i-1} - n_{i+1} & \text{if } i < l \\ n_{l-1} + n_l & \text{if } i = l \\ n_{i+1} - n_{i-1} & \text{if } i > l. \end{cases} \quad (8)$$

From (??) and (??) we obtain that  $THT$  is asymptotically normal with expectation  $\mu_T$  and variance  $\sigma_T^2$ .

5. For the  $PT$ -statistic we obtain ( $N_0 = 0$ ) after some straightforward but somewhat tedious calculations

$$\begin{aligned}
PT &\sim \sum_{i=1}^{l-1} (N_{i-1} + N_i - N_l) T_{i,N} + (N - N_l + N_{l-1}) T_{l,N} + \\
&\quad \sum_{i=l+1}^c (N - N_i - N_{i-1} + N_{l-1}) T_{i,N} \\
&\sim \sum_{i=1}^c \alpha_{P,i} W_i^*
\end{aligned}$$

with

$$\alpha_{P,i} = \begin{cases} N_{i-1} + N_i - N_l & \text{if } i < l \\ N - N_l + N_{l-1} & \text{if } i = l \\ N - N_i - N_{i-1} + N_{l-1} & \text{if } i > l \end{cases} \quad (9)$$

Again from (??) and (??) we obtain that  $PT$  is asymptotically normal with expectation  $\mu_P$  and variance  $\sigma_P^2$ .

6. Define for  $i = 2, \dots, l$

$$\mu_{(1\dots i-1)i} := N^{-1/2} n_i \left[ - \sum_{k=1}^{i-1} \theta_k n_k + \theta_i N_{i-1} \right] \cdot d(f, g).$$

The statistics  $S_{(1\dots i-1)i}$  are asymptotically normal with expectations  $\Delta\mu_{(1\dots i-1)i}$  and variances

$$\sigma_{(1\dots i-1)i}^2 = n_i N_{i-1} N_i I(g)$$

(cf. Büning and Kössler, 1999).

The linear combination of the  $W_j^*, j = 1, \dots, c$ ,

$$L_i = N_{i-1} W_i^* - \sum_{k=1}^{i-1} n_i W_k^*$$

is asymptotically normal with the same expectation  $\Delta\mu_{(1\dots i-1)i}$  and variance  $\sigma_{(1\dots i-1)i}^2$ , i.e.  $S_{(1\dots i-1)i}$  and  $L_i$  are asymptotically equivalent.

Analogously we have

$$S_{(c\dots i+1)i} \sim L'_i = (N - N_i) W_i^* - \sum_{k=i+1}^c n_i W_k^*.$$

Hence

$$\begin{aligned} MWT &\sim \sum_{i=2}^l (N_{i-1} W_i^* - \sum_{k=1}^{i-1} n_i W_k^*) + \sum_{i=l}^{c-1} ((N - N_i) W_i^* - \sum_{k=i}^c n_i W_k^*) \\ &= \sum_{i=1}^c \alpha_{P,i} W_i^* \end{aligned}$$

is asymptotically normal with expectation  $\mu_M$  and variance  $\sigma_M^2$ , where the coefficients  $\alpha_{P,i}$  are given by (??). ■

Denote  $\Lambda_i = \sum_{j=1}^i \lambda_j$ ,  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_c)$  and let  $\Phi$  be the c.d.f and  $z_{1-\alpha}$  the  $(1 - \alpha)$ -quantile of standard normal distribution.

The asymptotic power functions of all the tests considered have the form

$$\beta_{\boldsymbol{\lambda}, \boldsymbol{\theta}}(\Delta) = 1 - \Phi(z_{1-\alpha} - \Delta A(\boldsymbol{\lambda}, \boldsymbol{\theta}) C(f, g)),$$

i.e. power comparisons can be made in terms of the asymptotic efficacies  $A^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) \cdot C^2(f, g)$  or simply in terms of the  $A^2(\boldsymbol{\lambda}, \boldsymbol{\theta})$ . Denoting the corresponding terms by  $A_M^2$  for *MWT*,  $A_P^2$  for *PT*,  $A_T^2$  for *THT* and  $A_H^2$  for *HNT* then we have

$$\begin{aligned}
A_M^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) &= \frac{\left( \sum_{i=2}^l \theta_i \lambda_i (\Lambda_{i-1} + \Lambda_i - \Lambda_l) + \sum_{i=l}^c \theta_i \lambda_i (1 - \Lambda_i + \Lambda_{l-1} - \Lambda_{i-1}) \right)^2}{\frac{1}{3}(\Lambda_l^3 - \sum_{i=1}^c \lambda_i^3 - \lambda_l^3 + (1 - \Lambda_{l-1})^3) + 2\Lambda_{l-1}\lambda_l(1 - \Lambda_l)} \\
&= A_P^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) \\
A_T^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) &= \frac{\left( \sum_{i=1}^{l-1} (\theta_{i+1} - \theta_i) \lambda_i \lambda_{i+1} + \sum_{i=l}^{c-1} (\theta_i - \theta_{i+1}) \lambda_i \lambda_{i+1} \right)^2}{\sum_{i=1}^{c-2} \lambda_i \lambda_{i+1} (\lambda_i + \lambda_{i+1} - 2\lambda_{i+2}) + \lambda_{c-1} \lambda_c (\lambda_{c-1} + \lambda_c) + 4\lambda_{l-1} \lambda_l \lambda_{l+1}} \\
A_H^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) &= \frac{(\sum_{i=1}^c \lambda_i v_i \theta_i)^2}{\sum_{i=1}^c \lambda_i v_i^2}.
\end{aligned}$$

*Remark:* The MWT-test and the PT-test are *asymptotically* equivalent. Since the MWT-test is easier to perform the PT-test is not considered in the following. Note that the MWT-test and the PT-test are generally *not exactly* equivalent. If the MWT-test and the PT-test are based on the Wilcoxon scores then, however, the tests are equivalent.

Under the assumptions **(A)** the ARE of the tests MWT and THT, for example, is given by

$$ARE(MWT, THT) = \frac{A_M^2(\boldsymbol{\lambda}, \boldsymbol{\theta})}{A_T^2(\boldsymbol{\lambda}, \boldsymbol{\theta})}$$

(cf. Hájek and Šidák, 1967, ch.7.2.1, or Noether, 1955). For the case of equal sample sizes some asymptotic efficacies and ARE are computed in the next section.

Note that the results can easily be extended to the cases  $l = 1$  or  $l = c$  if we introduce the conventions  $N_0 = n_o = n_{c+1} = 0$ ,  $\Lambda_0 = \lambda_0 = \lambda_{c+1} = 0$ ,  $U_{0,1} = U_{c,c+1} = 0$  and  $\sum_{i=\iota}^{\kappa} z_i = 0$  if  $\kappa < \iota$  for arbitrarily chosen  $z_i$ .

TABLE I: The terms  $B_M, B_T, B_H$  and ARE in the case of equal sample sizes and equally spaced alternatives for  $c = 3, 4, 5, 6$

c	l	$B_M$	$B_T$	$B_H$	ARE(MWT,THT)	ARE(MWT,HNT)
3	2	2/9	2/9	2/9	1	1
4	2	25/56	3/8	1/2	1.19	0.89
4	3	25/56	3/8	1/2	1.19	0.89
5	2	121/140	8/15	26/25	1.62	0.83
5	3	8/15	8/15	14/25	1	0.95
5	4	121/140	8/15	26/25	1.62	0.83
6	2	147/100	25/36	65/36	2.12	0.81
6	3	49/60	25/36	11/12	1.18	0.89
6	4	49/60	25/36	11/12	1.18	0.89
6	5	147/100	25/36	65/36	2.12	0.81

### 2.3 The Special Case of Equal Sample Sizes

In the special case of equal sample sizes,  $n_1 = \dots = n_c$ , we have  $\boldsymbol{\lambda} = (1, \dots, 1)/c$  and therefore

$$A_M^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \frac{3 \left[ \sum_{i=2}^l \theta_i(2i-1-l) + \sum_{i=l}^c \theta_i(c-2i+l) \right]^2}{c[l(l^2-1) + (c-l+1)((c-l+1)^2-1) + 6(l-1)(c-l)]}$$

and

$$A_T^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \frac{(2\theta_l - \theta_c)^2}{6c}.$$

If  $c = 3$  and  $l = 2$  we obtain  $A_M^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) = A_T^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) = A_H^2(\boldsymbol{\lambda}, \boldsymbol{\theta}) = (2\theta_2 - \theta_3)^2/18$ , i.e. the ARE of these three tests is 1. (This fact is evident for MWT and THT, since the statistics are the same.) For  $c = 3, 4, 5, 6$  tables I and II contain ARE-values for equally spaced alternatives and some other types of alternatives, respectively. For equally spaced alternatives,  $\delta := |\theta_{i+1} - \theta_i|$ ,  $i = 1, \dots, c-1$ , the asymptotic efficacies have the form

$$K(\delta) = B \cdot \delta^2 \cdot C^2(f, g),$$

where  $K$  and  $B$  stand for  $K_M, B_M$  or  $K_T, B_T$  or  $K_H, B_H$ . Some values for  $B$  are also presented in table I. In this special case the HNT-test is the asymptotically most powerful one, MWT is second and THT is third best.

TABLE II: The ARE(MWT,THT) and ARE(MWT,HNT) for some types of alternatives in the case of equal sample sizes

c	l	type of alternative	ARE(MWT,THT)	ARE(MWT,HNT)
3	2	arbitrary	1	1
4	2	$\theta_1 < \theta_2 = \theta_3 = \theta_4$	0.43	$\infty$
		$\theta_1 = \theta_2 > \theta_3 = \theta_4$	1.71	0.57
		$\theta_1 = \theta_2 = \theta_3 > \theta_4$	1.71	0.57
5	2	$\theta_1 < \theta_2 = \theta_3 = \theta_4 = \theta_5$	0.21	1.16
		$\theta_1 = \theta_2 > \theta_3 = \theta_4 = \theta_5$	1.93	0.52
		$\theta_1 = \theta_2 = \theta_3 > \theta_4 = \theta_5$	3.43	0.61
		$\theta_1 = \theta_2 = \theta_3 = \theta_4 > \theta_5$	1.93	0.65
5	3	$\theta_1 < \theta_2 = \theta_3 = \theta_4 = \theta_5$	1	0.73
		$\theta_1 = \theta_2 < \theta_3 = \theta_4 = \theta_5$	1	1.30
		$\theta_1 = \theta_2 = \theta_3 > \theta_4 = \theta_5$	1	1.30
		$\theta_1 = \theta_2 = \theta_3 = \theta_4 > \theta_5$	1	0.73
6	2	$\theta_1 < \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6$	0.12	0.31
		$\theta_1 = \theta_2 > \theta_3 = \theta_4 = \theta_5 = \theta_6$	1.92	0.49
		$\theta_1 = \theta_2 = \theta_3 > \theta_4 = \theta_5 = \theta_6$	4.32	0.64
		$\theta_1 = \theta_2 = \theta_3 = \theta_4 > \theta_5 = \theta_6$	4.32	0.70
		$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 > \theta_6$	1.92	0.74
6	3	$\theta_1 < \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6$	0.6	2.2
		$\theta_1 = \theta_2 < \theta_3 = \theta_4 = \theta_5 = \theta_6$	0.6	$\infty$
		$\theta_1 = \theta_2 = \theta_3 > \theta_4 = \theta_5 = \theta_6$	1.35	0.55
		$\theta_1 = \theta_2 = \theta_3 = \theta_4 > \theta_5 = \theta_6$	2.4	0.55
		$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 > \theta_6$	1.35	0.55

In table II some other alternatives are considered. This table contains only the case  $l \leq c/2$ . Entries for  $l > c/2$  can be obtained by considering the symmetric case, for example:  $c = 4, l = 2, \theta_1 < \theta_2 = \theta_4$  is symmetric to  $c = 4, l = 3, \theta_1 = \theta_3 > \theta_4$ . From table II we see that in most cases the HNT-test has the highest efficacy. But there are also types of alternatives for which the THT-test performs best and the HNT-test is bad. The entry  $\infty$  denotes that the efficacy of the HNT-test is zero for that alternatives. The MWT-test seems to be a good compromise over the whole range of alternatives.

*Remarks:* 1. In the case of ordered alternatives simulation studies based on various sample sizes show that for  $n_i = 40, i = 1, \dots, c$ , the power of the considered tests is generally well approximated by their asymptotic power function. However, the approximation often works well even for  $n_i = 10, i = 1, \dots, c$ , as for the normal and double exponential (cf. Büning and Kössler, 1999). We conjecture that similar results can be obtained for the types of tests considered here.

2. For equal sample sizes and for the general alternative  $H_{1A}$  the MWT-test seems to be preferred. For unequal moderate up to large sample sizes the efficacies of the three tests (MWT, THT and HNT) can be computed for the extreme cases (one difference  $\theta_{i+1} - \theta_i \neq 0$ , the other differences zero) and possibly for the equispaced case. Then it might be that a decision for one of the tests can be made.

### 3. GENERALIZED MWT- AND THT- STATISTICS

Consider now the test problem  $(H_0, H_{1B})$ . The generalized Mack-Wolfe type and the generalized Tryon-Hettmansperger-type statistics are given by

$$\begin{aligned} GMWT &= \sum_{i=1}^{l-1} \omega_{Mi} S_{(1\dots i)i+1} + \sum_{i=l}^{c-1} \omega_{Mi} S_{(c\dots i+1)i} \\ GTHT &= \sum_{i=1}^{l-1} \omega_{Ti} U_{i,i+1} + \sum_{i=l}^{c-1} \omega_{Ti} U_{i+1,i} \end{aligned}$$

with the weight vectors  $\boldsymbol{\omega}_M = (\omega_{M1}, \dots, \omega_{Mc-1})$  and  $\boldsymbol{\omega}_T = (\omega_{T1}, \dots, \omega_{Tc-1})$ ,

respectively. Now, we determine the weights in such a way that the efficacies of the tests based on *GMWT* and *GTHT* become maximal.

Let be  $\boldsymbol{\eta}_M = (\eta_{M1}, \dots, \eta_{Mc-1})$  and  $\boldsymbol{\eta}_T = (\eta_{T1}, \dots, \eta_{Tc-1})$ , where

$$\begin{aligned}\eta_{Mi} &= \begin{cases} \frac{d}{d\Delta} E_{H_1}(S_{(1\dots i)i+1})|_{\Delta=0} & \text{if } i < l \\ \frac{d}{d\Delta} E_{H_1}(S_{(c\dots i+1)i})|_{\Delta=0} & \text{if } i \geq l, \end{cases} \\ \eta_{Ti} &= \begin{cases} \frac{d}{d\Delta} E_{H_1}(U_{i,i+1})|_{\Delta=0} & \text{if } i < l \\ \frac{d}{d\Delta} E_{H_1}(U_{i+1,i})|_{\Delta=0} & \text{if } i \geq l \end{cases}\end{aligned}$$

and  $E_{H_1}$  denotes the expectation under  $H_{1B}$ . The covariance matrices of the substatistics  $S_{(1)2}, \dots, S_{(1\dots l-1)l}, S_{(c\dots l+1)l}, \dots, S_{c(c-1)}$  and  $U_{1,2}, \dots, U_{l-1,l}, U_{l+1,l}, \dots, U_{c,c-1}$ , are denoted by  $\boldsymbol{\Sigma}_M$  and  $\boldsymbol{\Sigma}_T$ , respectively.

Let  $\boldsymbol{\eta}$ ,  $\boldsymbol{\omega}$  and  $\boldsymbol{\Sigma}$  be written for  $\boldsymbol{\eta}_M$ ,  $\boldsymbol{\omega}_M$ ,  $\boldsymbol{\Sigma}_M$  or  $\boldsymbol{\eta}_T$ ,  $\boldsymbol{\omega}_T$ ,  $\boldsymbol{\Sigma}_T$ . Then the efficacy is given by

$$K_{\mathbf{n}} = \frac{(\boldsymbol{\omega}'\boldsymbol{\eta})^2}{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}}$$

where  $K_{\mathbf{n}}$  may be  $K_{M,\mathbf{n}}$  or  $K_{T,\mathbf{n}}$ . Assume, that  $\boldsymbol{\Sigma}$  is a regular  $(c-1, c-1)$  matrix, then the efficacy becomes maximal if

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{opt} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} \quad (10)$$

(cf. Rao 1966, ch. 1f.1). Given the vectors  $\mathbf{n}$  and  $\boldsymbol{\theta}$ , the maximal achievable efficacy is

$$K_{\mathbf{n}}^{max} = \boldsymbol{\eta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta}.$$

In the following theorem it is shown that  $K_{\mathbf{n}}^{max}$  is the same for the tests based on *GMWT* and *GTHT*, each test statistic with its optimal weights.

For simplicity, denote the spacings of the alternative by  $\delta_i := \theta_{i+1} - \theta_i$ ,  $i = 1, \dots, c-1$  and recall that  $C^2(f, g) = d^2(f, g)/I(g)$ .

**THEOREM 2:** The maximal achievable efficacy is the same for the tests based on *GMWT* and *GTHT*, and it is given by

$$\begin{aligned}\boldsymbol{\eta}'_M \boldsymbol{\Sigma}_M^{-1} \boldsymbol{\eta}_M &= \boldsymbol{\eta}'_T \boldsymbol{\Sigma}_T^{-1} \boldsymbol{\eta}_T = \\ C^2(f, g) &\cdot \left[ \frac{2}{N^2} \sum_{i < j} \delta_i \delta_j N_i (N - N_j) + \frac{1}{N^2} \sum_{i=1}^{c-1} \delta_i^2 N_i (N - N_i) \right].\end{aligned}$$

*Proof:* 1. For the Mack-Wolfe-type test the asymptotic covariance matrix  $\Sigma_M = (\sigma_{M,ij})_{i,j=1,\dots,c-1}$  is given by

$$\sigma_{M,ij} = \begin{cases} \sigma_{(1\dots i)i+1}^2 & \text{if } i = j < l \\ \sigma_{(c\dots i+1)i}^2 & \text{if } i = j \geq l \\ \rho & \text{if } i = l-1, j = l \text{ or } i = l, j = l-1 \\ 0 & \text{else,} \end{cases}$$

where

$$\begin{aligned} \sigma_{(1\dots i)i+1}^2 &:= N_i N_{i+1} n_{i+1} I(g) \\ \sigma_{(c\dots i+1)i}^2 &:= (N - N_{i-1})(N - N_i) n_i I(g) \\ \rho &:= N_{l-1}(N - N_l) n_l I(g) = cov_{asy}(S_{(1\dots l-1)l}, S_{(c\dots l+1)l}) \end{aligned}$$

(cf. e.g. Puri, 1965). The inverse matrix  $\Sigma_M^{-1} = (\sigma_M^{ij})_{i,j=1,\dots,c-1}$  is given by

$$\sigma_M^{ij} = \begin{cases} \frac{1}{\sigma_{(1\dots i)i+1}^2} & \text{if } i = j < l-1 \\ \frac{1}{\sigma_{(c\dots i+1)i}^2} & \text{if } i = j \geq l+1 \\ \frac{d_l}{D} & \text{if } i = j = l-1 \\ \frac{d_{l-1}}{D} & \text{if } i = j = l \\ -\frac{\rho}{D} & \text{if } i = l-1, j = l \text{ or } i = l, j = l-1 \\ 0 & \text{else,} \end{cases}$$

where

$$\begin{aligned} D &= d_{l-1} d_l - \rho^2 = N n_l^3 N_{l-1} (N - N_l) I(g) \\ d_{l-1} &= N_{l-1} N_l n_l I(g) \\ d_l &= (N - N_{l-1})(N - N_l) n_l I(g). \end{aligned}$$



The components  $\eta_{Mi}, i = 1, \dots, c-1$ , of  $\boldsymbol{\eta}_M$  are given by

$$\begin{aligned}
\eta_{Mi} &= N^{-1/2} n_{i+1} \left[ - \sum_{k=1}^i \theta_k n_k + \theta_{i+1} N_i \right] \cdot d(f, g) \\
&= N^{-1/2} n_{i+1} \sum_{k=1}^i \delta_k N_k \cdot d(f, g) \\
&=: \mu_{(1\dots i)i+1} \quad \text{if } i \leq l-1 \quad \text{and} \\
\eta_{Mi} &= N^{-1/2} n_i \left[ - \sum_{k=i+1}^c \theta_k n_k + \theta_i (N - N_i) \right] \cdot d(f, g) \\
&= -N^{-1/2} n_i \sum_{k=i}^{c-1} \delta_k (N - N_k) \cdot d(f, g) \\
&=: \mu_{(c\dots i+1)i} \quad \text{if } i \geq l
\end{aligned}$$

With the optimal weights  $\omega_{Mi}^{opt}$  from (??) ,

$$\omega_{Mi}^{opt} = \begin{cases} \frac{\mu_{(1\dots i)i+1}}{\sigma_{(1\dots i)i+1}^2} & \text{if } i < l-1 \\ \frac{\mu_{(c\dots i+1)i}}{\sigma_{(c\dots i+1)i}^2} & \text{if } i > l \\ \frac{d_l}{D} \mu_{(1\dots l-1)l} - \frac{\rho}{D} \mu_{(c\dots l+1)l} & \text{if } i = l-1 \\ \frac{d_{l-1}}{D} \mu_{(c\dots l+1)l} - \frac{\rho}{D} \mu_{(1\dots l-1)l} & \text{if } i = l \end{cases}$$

the efficacy becomes

$$\begin{aligned}
K_{M,\mathbf{n}}^{max} &= \sum_{i=1}^{l-2} \frac{\mu_{(1\dots i)i+1}^2}{\sigma_{(1\dots i)i+1}^2} + \sum_{i=l+1}^{c-1} \frac{\mu_{(c\dots i+1)i}^2}{\sigma_{(c\dots i+1)i}^2} + \frac{d_l}{D} \mu_{(1\dots l-1)l}^2 + \frac{d_{l-1}}{D} \mu_{(c\dots l+1)l}^2 \\
&\quad - 2 \frac{\rho}{D} \mu_{(1\dots l-1)l} \mu_{(c\dots l+1)l} \\
&= \frac{1}{N} \sum_{i=1}^{l-2} \frac{n_{i+1}}{N_{i+1} N_i} \left[ \sum_{k=1}^i \delta_k N_k \right]^2 \cdot C^2(f, g) \\
&\quad + \frac{1}{N} \sum_{i=l+1}^{c-1} \frac{n_i}{(N - N_{i-1})(N - N_i)} \left[ \sum_{k=i}^{c-1} \delta_k (N - N_k) \right]^2 \cdot C^2(f, g) +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{N^2} \frac{N - N_{l-1}}{N_{l-1}} \left[ \sum_{k=1}^{l-1} \delta_k N_k \right]^2 \cdot C^2(f, g) \\
& + \frac{1}{N^2} \frac{N_l}{N - N_l} \left[ \sum_{k=l}^{c-1} \delta_k (N - N_k) \right]^2 \cdot C^2(f, g) \\
& + 2 \frac{1}{N^2} \sum_{k=1}^{l-1} \sum_{j=l}^{c-1} \delta_k \delta_j N_k (N - N_j) \cdot C^2(f, g) \\
= & \frac{1}{N N_{l-1}} \left[ \sum_{i=1}^{l-2} \delta_i^2 N_i (N_{l-1} - N_i) + 2 \sum_{i,j=1, i < j}^{l-2} \delta_i \delta_j N_i (N_{l-1} - N_j) \right] C^2(f, g) \\
& + \frac{1}{N(N - N_l)} \left[ \sum_{i=l}^{c-1} \delta_i^2 N_i (N - N_i) + 2 \sum_{i,j=l, i < j}^{c-1} \delta_i \delta_j N_i (N - N_j) \right] C^2(f, g) \\
& + \frac{1}{N^2} \frac{N - N_{l-1}}{N_{l-1}} \left[ \sum_{k=1}^{l-1} \delta_k N_k \right]^2 C^2(f, g) \\
& + \frac{1}{N^2} \frac{N_l}{N - N_l} \left[ \sum_{k=l}^{c-1} \delta_k (N - N_k) \right]^2 C^2(f, g) \\
& + \frac{2}{N^2} \sum_{k=1}^{l-1} \sum_{j=l}^{c-1} \delta_k \delta_j N_k (N - N_j) C^2(f, g),
\end{aligned}$$

where the identity  $\sum_{i=\iota}^{\kappa} n_i / (N_{i-1} N_i) = (N_{\kappa} - N_{\iota-1}) / (N_{\kappa} N_{\iota-1})$ ,  $1 < \iota \leq \kappa \leq c$ , is used. Summarizing the terms of the first and the third row and also the terms of the second and fourth row we obtain

$$\begin{aligned}
K_{M, \mathbf{n}}^{max} &= \frac{1}{N^2} \left[ \sum_{i=1}^{l-1} \delta_i^2 N_i (N - N_i) + 2 \sum_{i,j=1, i < j}^{l-1} \delta_i \delta_j N_i (N - N_j) \right] C^2(f, g) + \\
& + \frac{1}{N^2} \left[ \sum_{i=l}^{c-1} \delta_i^2 N_i (N - N_i) + 2 \sum_{i,j=l, i < j}^{c-1} \delta_i \delta_j N_i (N - N_j) \right] C^2(f, g) + \\
& + \frac{2}{N^2} \sum_{k=1}^{l-1} \sum_{j=l}^{c-1} \delta_k \delta_j N_k (N - N_j) C^2(f, g) \\
& = \left[ \frac{2}{N^2} \sum_{i < j} \delta_i \delta_j N_i (N - N_j) + \frac{1}{N^2} \sum_{i=1}^{c-1} \delta_i^2 N_i (N - N_i) \right] C^2(f, g).
\end{aligned}$$

2. For the Tryon-Hettmansperger-type test statistics the asymptotic covariance matrix  $\Sigma_T = (\sigma_{T,ij})_{i,j=1,\dots,c-1}$  is given by (see also Puri, 1965):

$$\sigma_{T,ij} = I(g) \cdot \begin{cases} n_i n_{i+1} (n_i + n_{i+1}) & \text{if } j = i \\ -n_i n_{i+1} n_{i+2} & \text{if } j = i + 1 \text{ and } i \neq l - 1 \\ -n_{i-1} n_i n_{i+1} & \text{if } j = i - 1 \text{ and } i \neq l \\ n_{l-1} n_l n_{l+1} & \text{if } j = i - 1 = l - 1 \text{ or } j = i + 1 = l \\ 0 & \text{else.} \end{cases}$$

In order to calculate optimal weights the inverse  $\Sigma_T^{-1} := (\sigma_T^{ij})_{i,j=1,\dots,c-1}$  of  $\Sigma_T$  is needed. This matrix can be obtained by using arguments of Fiedler (1972, Theorem 12.2) or by using the result for ordered alternatives (cf. Kössler and Büning, 1996) where only two entries in  $\Sigma$  are changed. Define

$$a_{ij} = \begin{cases} N_i(N - N_j) & \text{if } i \leq j < l \text{ or } l \leq i \leq j \\ N_j(N - N_i) & \text{if } i > j \geq l \text{ or } l > i > j \\ -N_i(N - N_j) & \text{if } i < l \leq j \\ -N_j(N - N_i) & \text{if } i \geq l > j \end{cases}$$

Then the entries  $\sigma_T^{ij}$  are given by

$$\sigma_T^{ij} = \frac{1}{I(g)} \cdot \frac{a_{ij}}{N n_i n_{i+1} n_j n_{j+1}}$$

Since  $\eta_{T,j} = N^{-1/2} n_j n_{j+1} |\delta_j|$  the optimal weights are proportional to

$$\omega_{T,i}^{opt} := \frac{1}{N^{3/2} n_i n_{i+1}} \sum_{j=1}^{c-1} a_{ij} |\delta_j|$$

and because of  $K_{T,\mathbf{n}}^{max} = \boldsymbol{\eta}_T' \Sigma_T^{-1} \boldsymbol{\eta}_T$  the maximal achievable efficacy becomes

$$K_{T,\mathbf{n}}^{max} = \left[ \frac{2}{N^2} \sum_{i < j} \delta_i \delta_j N_i (N - N_j) + \frac{1}{N^2} \sum_{i=1}^{c-1} \delta_i^2 N_i (N - N_i) \right] C^2(f, g). \blacksquare$$

*Remarks:*

1. a) In the special case of  $\delta_k \neq 0, \delta_j = 0$  for  $j \neq k$ , optimal weights for the THT-test can easily be obtained by

$$\omega_{T,i}^{*,opt} = \frac{1}{n_i n_{i+1}} a_{ik}.$$

b) In the special case of  $|\delta_k| = \delta, k = 1, \dots, c$  and equal sample sizes optimal weights for the MWT-test are

$$\omega_{M_i}^{*,opt} := \frac{1}{2c} \begin{cases} c & \text{if } i < l-1 \text{ or } i > l \\ (c-l+1)(2l-c) & \text{if } i = l-1 \\ l(c-2l+2) & \text{if } i = l \end{cases}$$

The formulas for the optimal weights of the MWT-test in the special case 1.a) as well as for the THT-test in the special case 1.b) are more complex and therefore omitted here.

2. Generalized HNT-tests (GHNT-tests) can be defined in a similar way. In the light of theorem 1 it can be seen that to each weight vector  $\omega_T$  of the GTHT-test there exists a weight vector  $\omega_H$  of the GHNT-test and vice versa so that these tests are asymptotically equivalent. Therefore the maximal achievable efficacy for the GHNT-test is the same as for the GTHT-test.

#### 4. THE CASE OF UNKNOWN PEAKS

Assume now that the alternative is given by  $H_{1C}$ , i.e the peak  $l$  is unknown and it has to be estimated from the data. We follow the proposal of Chen and Wolfe (1990) and Chen (1991) who consider only Mann-Whitney scores. For the balanced design, i.e.  $n_1 = \dots = n_c$ , they determine asymptotical critical values of the statistic

$$MW_{max} := \max_{l=1, \dots, c} MW_l^*$$

where

$$MW_l^* = \frac{MW_l - E(MW_l)}{(Var(MW_l))^{1/2}}$$

are the standardized Mack-Wolfe statistics with a given peak  $l$ . Now we apply this idea to the Mack-Wolfe-type statistics with arbitrarily chosen scores, cf. section 2.1.

Let  $MWT_l$  be the Mack-Wolfe type statistic with peak  $l$  and  $MWT_l^*$  the standardized statistics

$$MWT_l^* := \frac{MWT_l}{\sigma_{M,l}},$$

where  $\sigma_{M,l}^2$  is given in 2.2. The cases  $l = 1$  and  $l = c$  also make sense since one of the sums in the  $MWT$ -statistic vanishes. The formula for  $\sigma_{M,l}^2$  remains valid if we set  $N_0 = 0$ . Then the peak  $l$  is estimated by

$$\hat{l} := \arg \max_{l=1,\dots,c} MWT_l^*$$

and the statistic

$$CWT := \max_{l=1,\dots,c} MWT_l^*$$

has, under  $H_0$ , the same asymptotic c.d.f. as the  $MW_{max}$ -statistic. The vector  $\mathbf{MWT}^* := (MWT_1^*, \dots, MWT_c^*)$  is, under  $H_0$ , asymptotically normally distributed with expectation vector equal to zero and a covariance matrix which depends only on the sample sizes  $n_i$  (cf. Chen, 1991). For equal sample sizes the critical values of Chen (1991) can be used.

Let be  $\mathbf{x} = (x_1, \dots, x_{c-1})^T$  and  $\Sigma_M^*$  the covariance matrix of the statistic  $\mathbf{MWT}^*$  which is assumed to be regular. Then, for  $\tau > 0$ ,

$$P_0(\max_{l=1,\dots,c-1} MWT_l^* < \tau) = \int_{-\infty}^{\tau} \cdots \int_{-\infty}^{\tau} \int_{-\tau}^{\tau} \frac{\det(\Sigma_M^*)^{-1/2}}{(2\pi)^{(c-1)/2}} \exp(-\frac{1}{2} \mathbf{x}^T \Sigma_M^{*-1} \mathbf{x}) d\mathbf{x}.$$

Let  $\tau_{1-\alpha}$  be the  $(1 - \alpha)$ -quantile of this c.d.f.  $H_0$  is rejected in favour of  $H_{1C}$  if  $CWT > \tau_{1-\alpha}$ . This test is called Chen-Wolfe-type test (CWT-test).

Denote by  $\mu_{M,l}$  the  $l$ th component of the expectation vector of  $\mathbf{MWT}^*$  which is given by (??) and let  $\boldsymbol{\mu} = (\mu_{M,1}, \dots, \mu_{M,c-1}) = \boldsymbol{\mu}(\Delta)$ . Then the asymptotic power function of the Chen-Wolfe-type test is given by

$$\beta_{\mathbf{n},\boldsymbol{\theta}}(\Delta) = 1 - \det(\Sigma_M^*)^{-1/2} (2\pi)^{-(c-1)/2} \cdot \int_{-\infty}^{\tau_{1-\alpha}} \cdots \int_{-\infty}^{\tau_{1-\alpha}} \int_{-\tau_{1-\alpha}}^{\tau_{1-\alpha}} \exp(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma_M^{*-1} (\mathbf{x} - \boldsymbol{\mu})) d\mathbf{x}. \quad (11)$$

To get an impression of the shape of this function we consider the case  $c = 3$  separately. The case  $c = 2$  is trivial since it boils down to comparing a two-sided power to a one-sided one.

In the case  $c = 3$  the statistics  $MWT_l$ ,  $l = 1, 2, 3$  are given by

$$\begin{aligned} MWT_1 &= S_{(32)1} + S_{(3)2} \\ MWT_2 &= S_{(1)2} + S_{(3)2} \\ MWT_3 &= S_{(12)3} + S_{(1)2} \sim -MWT_1 \end{aligned}$$

with the asymptotic variances

$$\sigma_1^2 := \sigma_{M,1}^2 = I(g) \left[ \frac{1}{3} (N^3 - \sum_{i=1}^3 n_i^3) \right] = \sigma_{M,3}^2 =: \sigma_3^2$$

and

$$\begin{aligned} \sigma_2^2 := \sigma_{M,2}^2 &= I(g) \left[ \frac{1}{3} (N_2^3 - n_1^3 - n_2^3 + (N - N_1)^3 - n_2^3 - n_3^3) + 2n_1n_2n_3 \right] \\ &= I(g)n_2(n_1^2 + n_1n_2 + n_2n_3 + n_3^2 + 2n_1n_3) \\ &= I(g)Nn_2(n_1 + n_3). \end{aligned}$$

The asymptotic covariances are given by

$$\text{cov}(MWT_1, MWT_2) \sim -\text{cov}(MWT_3, MWT_2) \sim I(g)Nn_2(n_3 - n_1).$$

Therefore,

$$\begin{aligned} \text{cov}(MWT_1^*, MWT_2^*) &\sim \left( \frac{3Nn_2}{(N^3 - \sum_{i=1}^3 n_i^3)(n_1 + n_3)} \right)^{1/2} (n_3 - n_1) \\ &\sim -\text{cov}(MWT_3^*, MWT_2^*). \end{aligned}$$

Given  $n_1, n_2, n_3$  the covariance matrix  $\Sigma_M^*$  is regular ( $n_1 \neq 0$ ) and the asymptotic power can be computed from (??) numerically. Here we only consider the case  $n_1 = n_3, n_2$  chosen arbitrarily.

The asymptotic null distribution of  $CWT$  can easily be calculated ( $\tau > 0$ ):

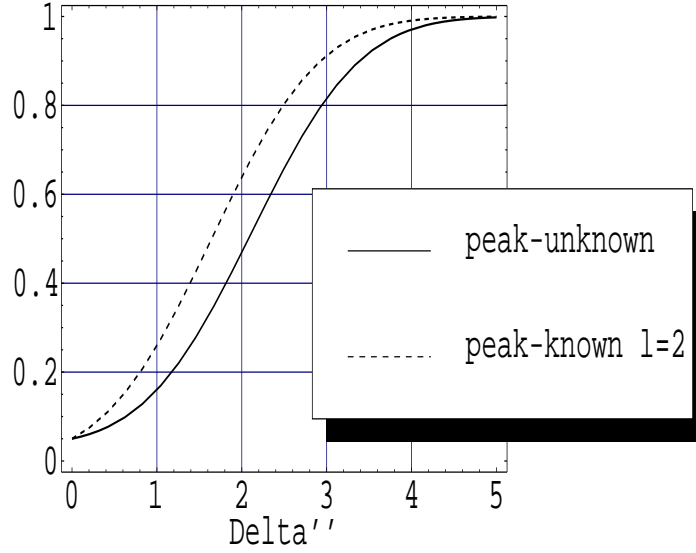
$$\begin{aligned} P_o(CWT < \tau) &= P_o(\max(|MWT_1|, MWT_2) < \tau) \\ &\sim \int_{-\tau}^{\tau} \int_{-\infty}^{\tau} \frac{1}{2\pi} e^{-\frac{1}{2}(t^2+u^2)} dt du \\ &= (\Phi(\tau) - \Phi(-\tau))\Phi(\tau). \end{aligned}$$

The  $(1 - \alpha)$ -quantile of this c.d.f can be computed numerically. For  $\alpha = 0.05$  we obtain  $\tau_{1-\alpha} = 2.123$  which coincides with the value calculated by Chen (1991) via simulation.

Recall that  $\delta_i = \theta_{i+1} - \theta_i, i = 1, 2$ . Under  $H_1$  we have

$$\begin{aligned} P_{\theta}(CWT < \tau_{1-\alpha}) &= P_{\theta}(\max(|MWT_1|, MWT_2) < \tau_{1-\alpha}) \sim \\ &\left( \Phi\left(\tau_{1-\alpha} - \frac{\mu_1}{\sigma_1}\right) - \Phi\left(-\tau_{1-\alpha} - \frac{\mu_1}{\sigma_1}\right) \right) \Phi\left(\tau_{1-\alpha} - \frac{\mu_2}{\sigma_2}\right) \end{aligned}$$

Figure 1: The asymptotic power of the MWT-test and CWT-test with known and unknown peaks, respectively, if  $c = 3, l = 2$



where

$$\begin{aligned}
 \mu_1 &:= E_{\boldsymbol{\theta}_{asy}}(MWT_1) = \frac{\Delta}{\sqrt{N}}(\delta_1 n_1 n_2 + \delta_2 n_2 n_3 + (\delta_1 + \delta_2) n_1 n_3) d(f, g) \\
 &= \frac{\Delta}{\sqrt{N}}(\delta_1 + \delta_2) n_1 (n_1 + n_2) d(f, g), \\
 \mu_2 &:= E_{\boldsymbol{\theta}_{asy}}(MWT_2) = \frac{\Delta}{\sqrt{N}}(\delta_1 n_1 n_2 - \delta_2 n_2 n_3) d(f, g) \\
 &= \frac{\Delta}{\sqrt{N}}(\delta_1 - \delta_2) n_1 n_2 d(f, g).
 \end{aligned}$$

Let  $\Delta' = \Delta(\delta_1 + \delta_2)(\lambda_1/2)^{1/2}C(f, g)$  and  $\Delta'' = \Delta(\delta_1 - \delta_2)((\lambda_1 \lambda_2)/2)^{1/2}C(f, g)$ . Then the asymptotic power of the CWT-test can be determined in terms of  $\Delta'$  and  $\Delta''$ :

$$\beta(\Delta', \Delta'') = 1 - (\Phi(\tau_{1-\alpha} - \Delta') - \Phi(-\tau_{1-\alpha} - \Delta')) \Phi(\tau_{1-\alpha} - \Delta'')$$

This asymptotic power has a lower bound which depends on only one variable, either  $\Delta'$  or  $\Delta''$ :

$$\beta(\Delta', \Delta'') \geq 1 - (\Phi(\tau_{1-\alpha} - \Delta') - \Phi(-\tau_{1-\alpha} - \Delta')) =: \gamma_1(\Delta')$$

and

$$\beta(\Delta', \Delta'') \geq 1 - (\Phi(\tau_{1-\alpha}) - \Phi(-\tau_{1-\alpha}))\Phi(\tau_{1-\alpha} - \Delta'') =: \gamma_2(\Delta'')$$

The lower bound in the last inequality is attained if  $\Delta' = 0$ , i.e.  $l = 2$  and  $|\delta_1| = |\delta_2|$ . Let us compare the peak-unknown asymptotic power  $\gamma_2$  with the peak-known asymptotic power, which is for  $l = 2$  given by

$$\beta_{\lambda, \theta}^M(\Delta'') = 1 - \Phi(z_{1-\alpha} - \Delta'').$$

Of course, the asymptotic power is larger if the peak of the umbrella is known, as illustrated by figure 1.

*Remark:* The idea of Chen and Wolfe (1990) and Chen (1991) can also be applied to *THT*- and *HNT*-statistics. For  $c = 3$  and  $n_1 = n_3$ ,  $n_2$  chosen arbitrarily, the tests based on the corresponding statistics  $\max_{l=1, \dots, c} THT$  and  $\max_{l=1, \dots, c} HNT$  have the same asymptotic power as the CWT-test. It can be shown that for unequal sample sizes,  $n_1 \neq n_3$ , the test based on  $\max_{l=1, \dots, c} HNT$  is always slightly better than the CWT-test and the CWT-test is always better than the test based on  $\max_{l=1, \dots, c} THT$ .

Another proposal for estimating the unknown peak  $l$  is given by Hothorn and Liese (1991).

## 5. OUTLOOK

The goal of this paper was threefold, first, to demonstrate, that for the test problem  $(H_0, H_{1A})$  the MWT-tests are also asymptotically serious competitors to the other existing tests, second to show, that for the test problem  $(H_0, H_{1B})$  the tests based on *GMWT* and *GTHT*, each test statistic with optimal weights, are asymptotically equivalent, and third, to compare the peak-known with the peak-unknown asymptotic power.

Analogously to the case of ordered alternatives the score generating function (1) can be chosen in such a way that the coefficient  $C(f, g)$  becomes large (Büning and Kössler 1999). Thus a test with comparable high power is obtained.

If the practising statistician has no exact information about the underlying distribution of his data an adaptive test can be applied which takes into



account the given data. Examples of such adaptive tests based on the concept of Hogg et.al. (1975) are given by Büning (1996) and Büning and Kössler (1998).

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