THE EFFICACY OF SOME c-SAMPLE RANK TESTS OF HOMOGENEITY AGAINST ORDERED ALTERNATIVES

Wolfgang Kössler* and Herbert Büning**

*Humboldt-Universität zu Berlin, Institut für Informatik,
D-10099 Berlin, Germany
**Freie Universität Berlin, Institut für Statistik und Ökonometrie,
D-14195 Berlin, Germany

For the c-sample location problem with ordered alternatives the test of Jonckheere (1954) and Terpstra (1952) is a well-known competitor to the parametric test introduced by Barlow et.al. (1972, p.184). Generalizations of the Jonckheere-Terpstra test are proposed e.g. by Puri (1965), Tryon and Hettmansperger (1973) and Büning and Kössler (1996). All these tests are based on the pairwise ranking method. In the present paper their efficacies are compared. The Jonckheere-type and Tryon-Hettmansperger-type test statistics can be further generalized by introducing weight coefficients to the substatistics. For the case of a specified alternative these weights are determined to obtain maximal efficacies. It is shown that the maximal achievable efficacies in the defined classes of generalized Jonckheere-type tests and Tryon-Hettmansperger-type tests always are equal.

KEY WORDS: Asymptotic Relative Efficiency, Jonckheere-Terpstra test, - Jonckheere-type test, Puri-type test, Tryon-Hettmansperger-type test.

1. INTRODUCTION

Let be $X_{i1}, \ldots, X_{in_i}, \quad i = 1, \cdots, c,$ independent random samples from a population with an absolutely continuous distribution function $F(x - \vartheta_i)$, $\vartheta_i \in \mathbf{R}$. In the following we assume that F is twice continuously differentiable on $(-\infty, \infty)$ except for a set of Lebesgue measure zero; f' denotes the derivative of the density f where it exists and it is defined to be zero, otherwise. Furthermore, f' is assumed to be bounded. We wish to test

$$H_0: \quad \vartheta_1 = \ldots = \vartheta_c$$

against one of the following alternatives

$H_{1A}: \quad \vartheta_1 \leq \ldots \leq \vartheta_c$ with $\vartheta_1 < \vartheta_c$ or

$H_{1B}: \quad \vartheta_1 \leq \ldots \leq \vartheta_c$ with $\vartheta_1 < \vartheta_c$ and specified $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_c)$.

The nonparametric test of Jonckheere (1954) and Terpstra (1952) is one of the most familiar tests for the treatment of these test problems. In section 2 some generalizations of the Jonckheere-Terpstra test are considered. Their efficacies depend on the underlying density f and the optimal score function ϕ as well as on the sample sizes and alternatives. In this paper the optimal score function is assumed to be fixed. Only the effect of the sample sizes and the alternatives on the efficacies and the asymptotic relative Pitman efficiencies (ARE) of different types of statistics is studied.

In section 3 the test statistics JT and TT are further generalized by introducing weight coefficients for the substatistics. For the test problem (H_0, H_{1B}) the weights can be determined in such a way that the efficacies become maximal. It is shown that the generalized versions of the tests JT and TT, each of them based on the "optimal" weights, always have the same efficacies and therefore they are asymptotically equivalent in the sense of ARE.

All the statistics considered are based on ranks of paired samples. Other types of rank statistics are obtained if the ranks are taken over all c samples. From Koziol and Reid (1977) we conclude that they are closely related to the statistics considered in this paper. Some types of such statistics are investigated by Fairley and Fligner (1987). The case of *equal sample sizes* was studied by Govindarajulu and Haller (1977) and Shirahata (1980). For the test problem (H_0, H_{1B}) Govindarajulu and Haller (1977) obtained optimal weights which yield the same maximal efficacy as we obtained for the tests considered here.

Kumar, Gill and Mehta (1994) and Kumar, Gill and Dhawan (1994) considered linear combinations of two-sample statistics where the medians of subsamples of odd size not less than 3 are ranked. Since the use of medians reduces the effect of extreme observations it is clear that their tests, called B_c and W_m, respectively, are suitable for long-tailed distributions. But for this situation a much simpler JT-test with appropriate scores, e.g. the JLT-test of Büning and Kössler (1996), is a very good competitor because for equal sample sizes and equally spaced θ-values we get $ARE(JLT, B_c) = 1.01$ if F is the doubleexponential and $ARE(JLT, B_c) = 1.07$ if F is the Cauchy distribution.

2. SOME TYPES OF RANK TESTS

We consider so called Jonckheere-type tests (see Büning and Kössler (1999)), Puri-type tests (cf. Puri (1965)) and Tryon-Hettmansperger-type tests (cf. Tryon and Hettmansperger (1973)). All these tests are generalizations of the Jonckheere-Terpstra test (see Jonckheere (1954) and Terpstra (1952)). The statistic of the Jonckheere-type test is defined by

$$JT = \sum_{i=2}^{c} S_{(1\ldots i-1)i}$$

with

$$S_{(1\ldots i-1)i} = N_i \cdot \sum_{k=N_{i-1}+1}^{N_i} a_{N_i}(R^k_{1\ldots i}),$$

$a_{N_i}(k) \in \mathbf{R}$, $N_i = \sum_{j=1}^{i} n_j$, $1 \le i \le c$ and $R^k_{1\ldots i}$ is the rank of the kth observation in the pooled i samples $X_{11}, \ldots, X_{1n_1}, \ldots, X_{i1}, \ldots, X_{in_i}$. $S_{(1\ldots i-1)i}$ is a two-sample linear rank statistic computed on the ith sample versus the combined data in the first $(i-1)$ samples.

The Puri-type test statistic is defined by

$$PT = \sum_{i<j} U_{i,j}$$

with

$$U_{i,j} = (n_i + n_j) \sum_{l=n_i+1}^{n_i+n_j} a_{n_i+n_j}(R^l_{ij}),$$

$a_{n_i+n_j}(l) \in \mathbf{R}$, $1 \le i, j \le c$, and R^l_{ij} is the rank of the lth observation in the combined two samples $X_{i1}, \ldots, X_{in_i}$ and $X_{j1}, \ldots, X_{jn_j}$.

The Tryon-Hettmansperger-type test statistic is given by

$$TT = \sum_{i=1}^{c-1} U_{i,i+1}$$

with $U_{i,j}$ defined as above.

The (exact or asymptotic) associated α-level tests reject H_0 in favour of H_1 (H_{1A} or H_{1B}) if JT, PT or TT are at least as large as the upper α-quantile of the (exact or asymptotic) null distribution of JT, PT or TT, respectively. For convenience the corresponding tests are called JT-test, PT-test and TT-test, respectively.

It is assumed that the scores $a_M(\cdot)$ in the definition of the test statistics above are generated by an absolutely continuous score function ϕ with $\lim_{M\to\infty} a_M(1+[uM]) = \phi(u), \quad 0 < u < 1$, $\phi \in L_2(0,1)$, where ϕ is associated with a density function g given by

$$\phi(u,g) := \phi(u) = -\frac{g'(G^{-1}(u))}{g(G^{-1}(u))}. \tag{1}$$

The function $\phi(u,g)$ is the so called optimal score function of the density function g with quantile function G^{-1} and it is assumed to be fixed throughout this paper. Define

$$d(f,g) = \int_0^1 \phi'(u,g)\cdot f(F^{-1}(u))\,du \qquad \text{and} \qquad I(g) = \int_0^1 \phi^2(u,g)\,du,$$

where $I(g)$ is the Fisher-Information of the density function g defined by (1), ϕ' represents the derivative of ϕ almost everywhere. It is assumed that $\int_0^1 \phi(u,g)\,du = 0$ and $I(g) < \infty$.

Note that the PT- and TT-statistics are special cases of the statistics T_N of Tryon and Hettmansperger (1973, cf. their equation (2)) whereas JT is generally not. In the latter statistic the ranks are taken over i samples whereas in the former one they are taken over two samples only. However, it can be shown, that the JT- statistics can *asymptotically* be described as a linear combination of Chernoff and Savage (1958) statistics. In this regard the present paper generalizes results of the paper of Tryon and Hettmansperger (1973) to unequal sample sizes.

Let be $\Delta > 0$, $\quad N = N_c$ and $\{(\vartheta_{1N}, \ldots, \vartheta_{cN})\}$ a sequence of "near" alternatives with $\sqrt{N}\vartheta_{iN}/\Delta = \theta_i$, $\quad \theta_1 \leq \ldots \leq \theta_c$, $\quad \theta_1 < \theta_c$. Denote

$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_c)$, $\mathbf{n}= (n_1, \ldots, n_c)$ and assume without loss of generality $\theta_1 = 0$. Under the further assumptions

$$min(n_1, \ldots, n_c) \longrightarrow \infty, \quad \frac{n_i}{N} \longrightarrow \lambda_i, \quad 0 < \lambda_i < 1, \quad i = 1, \ldots, c, \tag{2}$$

the test statistics JT, PT and TT are, under H_0 and H_1 (H_{1A} or H_{1B}), asymptotically normally distributed with some parameters $(0, \sigma_T^2)$ and - (μ_T, σ_T^2), respectively. Obviously, these parameters are also dependent on Δ, $\mathbf{n}$ and $\boldsymbol{\theta}$.

Recall that, under some regularity conditions, the efficacy of a statistic T is defined by the fraction $K_T = (\frac{\mu'_T(0)}{\sigma_T})^2$ with $\mu'_T(0) = \frac{d\mu_T(\Delta)}{d\Delta}|_{\Delta=0}$ (see Noether (1955)). Under the assumptions above the statistics JT, PT and TT satisfy the regularity conditions.

3. GENERALIZED JT- AND TT- STATISTICS

Consider now the test problem (H_0, H_{1B}). The generalized Jonckheere-type and the generalized Tryon-Hettmansperger-type statistics are given by

$$GJT = \sum_{i=2}^{c} \omega_{Ji} S_{(1 \ldots i-1)i}$$
$$GTT = \sum_{i=1}^{c-1} \omega_{Ti} U_{i,i+1}$$

with the weight vectors $\boldsymbol{\omega}_J = (\omega_{J2}, \ldots, \omega_{Jc})$ and $\boldsymbol{\omega}_T = (\omega_{T1}, \ldots, \omega_{Tc-1})$, respectively. Now, we determine the weights so that the efficacies of the tests based on GJT and GTT become maximal.

Let be $\boldsymbol{\eta}_J = (\eta_{J2}, \ldots, \eta_{Jc})$ and $\boldsymbol{\eta}_T = (\eta_{T1}, \ldots, \eta_{Tc-1})$, where

$$\eta_{Ji} = \frac{d}{d\Delta} E_{H_1}(S_{(1 \ldots i-1)i})|_{\Delta=0} \quad \text{and} \quad \eta_{Ti} = \frac{d}{d\Delta} E_{H_1}(U_{i,i+1})|_{\Delta=0}$$

and E_{H_1} denotes the expectation under H_1.

The covariance matrices of the substatistics $(S_{(1)2}, \ldots, S_{(1 \ldots c-1)c})$ and - $(U_{1,2}, \ldots, U_{c-1,c})$ are denoted by $\boldsymbol{\Sigma}_J$ and $\boldsymbol{\Sigma}_T$, respectively.

Let $\boldsymbol{\eta}$, $\boldsymbol{\omega}$ and $\boldsymbol{\Sigma}$ stand for $\boldsymbol{\eta}_J$, $\boldsymbol{\omega}_J$, $\boldsymbol{\Sigma}_J$ or $\boldsymbol{\eta}_T$, $\boldsymbol{\omega}_T$, $\boldsymbol{\Sigma}_T$. Then the efficacy is given by

$$K = \frac{(\boldsymbol{\omega}'\boldsymbol{\eta})^2}{\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}}. \qquad (3)$$

Assume, that $\boldsymbol{\Sigma}$ is a regular (c-1,c-1) matrix, then the efficacy becomes maximal if

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{opt} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\eta}$$

(see Rao (1966, p.48)). Given the vectors $\mathbf{n}$ and $\boldsymbol{\theta}$, the maximal achievable efficacy is

$$K^{max} = \boldsymbol{\eta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta}.$$

In the following theorem it is shown that K^{max} is the same for the tests based on GJT and GTT, each test with its optimal weights.

Theorem 1: The maximal achievable efficacy is the same for the tests based on GJT and GTT, i.e.

$$\begin{aligned}
\boldsymbol{\eta}_J'\boldsymbol{\Sigma}_J^{-1}\boldsymbol{\eta}_J &= \boldsymbol{\eta}_T'\boldsymbol{\Sigma}_T^{-1}\boldsymbol{\eta}_T \\
&= \frac{2}{N^2}\sum_{i<j}(\theta_{i+1}-\theta_i)(\theta_{j+1}-\theta_j)N_i(N-N_j)\cdot\frac{d^2(f,g)}{I(g)} + \\
&\quad \frac{1}{N^2}\sum_{i=1}^{c-1}(\theta_{i+1}-\theta_i)^2 N_i(N-N_i)\cdot\frac{d^2(f,g)}{I(g)}.
\end{aligned}$$

Proof: 1. For the Jonckheere-type test the covariance matrix is diagonal and the optimal weights are given by

$$\omega_{Ji}^{opt} = \frac{\mu_{(1\ldots i-1)i}}{\sigma^2_{(1\ldots i-1)i}},$$

$i = 2, \ldots, c$, where

$$\begin{aligned}
\mu_{(1\ldots i-1)i} &:= N^{-1/2}[-\sum_{k=1}^{i-1}\theta_k n_k n_i + \theta_i N_{i-1} n_i]\cdot d(f,g) \\
&= N^{-1/2} n_i \sum_{k=1}^{i-1}(\theta_{k+1}-\theta_k)N_k\cdot d(f,g) \qquad \text{and} \\
\sigma^2_{(1\ldots i-1)i} &:= N_i N_{i-1} n_i \cdot I(g)
\end{aligned}$$

(see Büning and Kössler (1996)). With the optimal weights ω_{Ji}^{opt} the efficacy becomes

$$\begin{aligned} K_J^{max} &= \sum_{i=2}^{c} \frac{\mu^2_{(1\ldots i-1)i}}{\sigma^2_{(1\ldots i-1)i}} \\ &= \frac{1}{N}\sum_{i=2}^{c}\frac{n_i}{N_{i-1}N_i}\left[\sum_{k=1}^{i-1}(\theta_{k+1}-\theta_k)N_k\right]^2 \cdot \frac{d^2(f,g)}{I(g)} \\ &= \frac{2}{N^2}\sum_{i<j}(\theta_{i+1}-\theta_i)(\theta_{j+1}-\theta_j)N_i(N-N_j)\cdot\frac{d^2(f,g)}{I(g)} + \\ &\quad + \frac{1}{N^2}\sum_{i=1}^{c-1}(\theta_{i+1}-\theta_i)^2 N_i(N-N_i)\cdot\frac{d^2(f,g)}{I(g)} \end{aligned}$$

after having rearranged the sums and by using the relation

$\sum_{i=l}^{c} \frac{n_i}{N_{i-1}N_i} = \frac{N-N_{l-1}}{N\cdot N_{l-1}}$ for $l = 2, \ldots, c$.

2. For the Tryon-Hettmansperger-type test statistics the covariance matrix $\mathbf{\Sigma}_T = (\sigma_{T_{i,j}})_{i,j=1,\ldots c-1}$ is given by (see also Puri (1965)):

$$\sigma_{T_{ij}} = I(g)\cdot\begin{cases} n_i n_{i+1}(n_i+n_{i+1}) & \text{if } j=i \\ -n_i n_{i+1} n_{i+2} & \text{if } j=i+1 \\ -n_{i-1} n_i n_{i+1} & \text{if } j=i-1 \\ 0 & \text{else.} \end{cases}$$

In order to calculate optimal weights the inverse $\mathbf{\Sigma}_T^{-1}$ of $\mathbf{\Sigma}_T$ is needed. $\mathbf{\Sigma}_T^{-1} := (\sigma_T^{ij})_{i,j=1,\ldots,c-1}$ can be obtained by using arguments of Fiedler (1972, Theorem 12.2) where

$$\sigma_T^{ij} = \frac{1}{I(g)}\cdot\frac{1}{N n_i n_{i+1} n_j n_{j+1}}\cdot\begin{cases} N_i(N-N_j) & \text{if } i \le j \\ N_j(N-N_i) & \text{else.} \end{cases}$$

Optimal weights are proportional to

$$\omega_{Ti}^{opt} := \frac{1}{N^{3/2} n_i n_{i+1}}\{\sum_{j=1}^{i} N_j(N-N_i)(\theta_{j+1}-\theta_j) + \sum_{j=i+1}^{c-1} N_i(N-N_j)(\theta_{j+1}-\theta_j)\}.$$

The maximal achievable efficacy becomes

$$\begin{aligned} K_T^{max} = \boldsymbol{\eta}_T' \boldsymbol{\Sigma}_T^{-1} \boldsymbol{\eta}_T &= \\ &= \frac{2}{N^2} \sum_{i<j} (\theta_{i+1} - \theta_i)(\theta_{j+1} - \theta_j) N_i (N - N_j) \cdot \frac{d^2(f,g)}{I(g)} + \\ &\quad + \frac{1}{N^2} \sum_{i=1}^{c-1} (\theta_{i+1} - \theta_i)^2 N_i (N - N_i) \cdot \frac{d^2(f,g)}{I(g)} \\ &= K_J^{max}. \blacksquare \end{aligned}$$

Corollary 1: The test statistics JT, PT and TT have the efficacies $K_{JT} = K_{PT} = A_{JT}(\mathbf{n}, \boldsymbol{\theta}) \cdot C(f,g)$ and $K_{TT} = A_{TT}(\mathbf{n}, \boldsymbol{\theta}) \cdot C(f,g)$, where

$$\begin{aligned} A_{JT}(\mathbf{n}, \boldsymbol{\theta}) &= \frac{3 \left[\sum_{k=2}^{c} \theta_k n_k (N_k + N_{k-1} - N)\right]^2}{N \cdot (N^3 - \sum_{i=1}^{c} n_i^3)} \\ A_{TT}(\mathbf{n}, \boldsymbol{\theta}) &= \frac{\left[\sum_{i=2}^{c-1} \theta_i n_i (n_{i-1} - n_{i+1}) + \theta_c n_{c-1} n_c\right]^2}{N \cdot \left[\sum_{i=1}^{c-2} n_i n_{i+1} (n_i + n_{i+1} - 2n_{i+2}) + n_{c-1} n_c (n_{c-1} + n_c)\right]} \end{aligned}$$

and $C(f,g) = d^2(f,g) \cdot [I(g)]^{-1}$.

Proof: The assertion for JT and TT follows easily from inserting $\boldsymbol{\omega} = (1, \ldots, 1)$ in (3). The rest follows from the asymptotic equivalence of JT and PT. $\blacksquare$

Corollary 2: In the special case of *equal sample sizes* $n_1 = \ldots n_c =: m$ the efficacies are given by

$$K_{JT} = K_{JT}(\boldsymbol{\theta}) = \frac{3 \cdot \left[\sum_{k=2}^{c} \theta_k (2k - 1 - c)\right]^2}{c^2 \cdot (c^2 - 1)} \cdot C(f,g)$$

and

$$K_{TT} = K_{TT}(\boldsymbol{\theta}) = \frac{\theta_c^2}{2c} \cdot C(f,g).$$

Corollary 3: In the special case of *equal sample sizes* and *equally spaced* $\boldsymbol{\theta}$-values, $\theta_{i+1} - \theta_i = \delta$, $i = 1, \ldots, c-1$, the efficacies are given by $K_{JT}(\boldsymbol{\theta}) = \frac{(c-1)(c+1)}{12} \delta^2 \cdot C(f,g)$ and $K_{TT}(\boldsymbol{\theta}) = \frac{(c-1)^2}{2c} \delta^2 \cdot C(f,g)$.

Corollary 4: (see Tryon and Hettmansperger (1973, Theorem 3.4)) In the special case of *equal sample sizes* and *equally spaced* $\boldsymbol{\theta}$-values optimal weights are proportional to $\omega_{Ji}^{opt} = 1$ and $\omega_{Ti}^{opt} = i \cdot (c - i)$.

In the last very special case we have $K_{JT} > K_{TT}$ if $c \geq 4$ and the JT-statistic is optimal in the class of the generalized JT- statistics.

Now, let us present some results on the asymptotic relative efficiency (ARE) of the tests JT and TT. Under the assumption (2) the ARE of the tests JT and TT is given by

$$ARE(JT,TT) = \lim_{N\to\infty} \frac{A_{JT}(\mathbf{n},\boldsymbol{\theta})}{A_{TT}(\mathbf{n},\boldsymbol{\theta})}$$

(see Noether (1955) and Hájek and Šidák (1967, ch.7.2.1)). The evaluation of $A_{JT}(\mathbf{n},\boldsymbol{\theta})$ and $A_{TT}(\mathbf{n},\boldsymbol{\theta})$ is generally rather difficult since there are subsets Θ', Θ'' of the parameter space $\Theta = \{\boldsymbol{\theta} : \theta_1 \leq \ldots \leq \theta_c, \theta_1 < \theta_c\}$ with $A_{JT}(\mathbf{n},\boldsymbol{\theta}) \geq A_{TT}(\mathbf{n},\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Theta'$ and $A_{JT}(\mathbf{n},\boldsymbol{\theta}) < A_{TT}(\mathbf{n},\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Theta''$. But if the alternative is restricted to some subset $\Theta_1 \subset \Theta$ with $\Theta_1 \subseteq \Theta'$ or $\Theta_1 \subseteq \Theta''$ just one of the inequalities above holds. In the case of *equal sample sizes* we obtain:

The $ARE(JT,TT)$ is maximal for $\theta_1 = \cdots = \theta_{\frac{c}{2}} < \theta_{\frac{c}{2}+1} = \cdots = \theta_c$ if c is even and for $\theta_1 = \cdots = \theta_{\frac{c-1}{2}} < \theta_{\frac{c+3}{2}} = \cdots = \theta_c$ if c is odd. It is then equal to

$$ARE^{max}(JT,TT) = \frac{3}{8(c^3-c)} \cdot \begin{cases} c^4 & \text{if c is even} \\ (c^2-1)^2 & \text{if c is odd.} \end{cases}$$

The $ARE(JT,TT)$ is minimal for $\theta_2 = \cdots = \theta_{c-1}, \theta_1 < \theta_c$ and it is then equal to

$$ARE^{min}(JT,TT) = \frac{6(c-1)}{c(c+1)}.$$

Table 1 represents ARE-values for some subsets of the parameter space Θ with $c = 4, 5$ and 6.

In most cases $ARE(JT,TT) > 1$ holds and the $ARE^{min}(JT,TT)$ is much closer to 1 than $ARE^{max}(JT,TT)$. Therefore, for the general alternative H_{1A}, the JT-test should be preferred.

Table 1: The ARE(JT,TT) in the case of equal sample sizes

c	subset of Θ		ARE(JT,TT)
4	$\boldsymbol{\theta}$ equally spaced		1.111
	$\theta_1 = \theta_2 \leq \theta_3 < \theta_4$,	$\theta_4 = \frac{\theta_3}{(\sqrt{10}-3)}$	1.000
	$\theta_1 = \theta_2 < \theta_3 \leq \theta_4$,	$\theta_4 = 2\theta_3$	1.225
	$\theta_1 \leq \theta_2 = \theta_3 \leq \theta_4$,	$\theta_1 < \theta_4$	0.900 (min.)
	$\theta_1 = \theta_2 < \theta_3 = \theta_4$		1.600 (max.)
5	$\boldsymbol{\theta}$ equally spaced		1.250
	$\theta_1 = \theta_2 \leq \theta_3 \leq \theta_4 < \theta_5$,	$\theta_5 = \frac{\theta_4}{\sqrt{5}-2}$	1.000
	$\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 = \theta_5$,	$\theta_5 = \frac{\theta_2}{3-\sqrt{5}}$	1.000
	$\theta_1 = \theta_2 \leq \theta_3 \leq \theta_4 < \theta_5$,	$\theta_5 = 2\theta_4$	1.250
	$\theta_1 \leq \theta_2 = \theta_3 = \theta_4 \leq \theta_5$,	$\theta_1 < \theta_5$	0.800 (min.)
	$\theta_1 = \theta_2 \leq \theta_3 \leq \theta_4 = \theta_5$,	$\theta_1 < \theta_5$	1.800 (max.)
6	$\boldsymbol{\theta}$ equally spaced		1.400
	$\theta_1 = \theta_2 \leq \theta_3 = \theta_4 \leq \theta_5 < \theta_6$,	$\theta_6 = \frac{3\theta_5}{\sqrt{35}-5}$	1.000
	$\theta_1 < \theta_2 \leq \theta_3 = \theta_4 \leq \theta_5 = \theta_6$,	$\theta_6 = \frac{3\theta_2}{8-\sqrt{35}}$	1.000
	$\theta_1 = \theta_2 \leq \theta_3 = \theta_4 \leq \theta_5 < \theta_6$,	$\theta_6 = 2\theta_5$	1.207
	$\theta_1 = \theta_2 \leq \theta_3 = \theta_4 \leq \theta_5 = \theta_6$	$\theta_1 < \theta_5$	1.829
	$\theta_1 \leq \theta_2 = \theta_3 = \theta_4 = \theta_5 \leq \theta_6$	$\theta_1 < \theta_6$	0.714 (min.)
	$\theta_1 = \theta_2 = \theta_3 < \theta_4 = \theta_5 = \theta_6$		2.314 (max.)

4. CONCLUSIONS

The goal of this paper was twofold, first of all, to show, that for the test problem (H_0, H_{1B}) the tests based on GJT and GTT, each with their optimal weights, are asymptotically equivalent, and second, that for the test problem (H_0, H_{1A}) the JT-test is a very good competitor to the other existing tests. For equal sample sizes and equally spaced alternatives all the tests, the JT-test, the GTT-test with the weights $i(c-i)$ and the test of Govindarajulu and Haller (1977) with appropriate weights are optimal in the sense of ARE.

If the choice of the score generating function (1) is based on f in such a way that the coefficient $C(f, g)$ becomes large (see Büning and Kössler (1996)) we obtain a test with comparable high power.

But usually the practising statistician has no exact information about the underlying distribution of his data. Thus an adaptive test should be applied which takes the given data into account. Examples of such adaptive tests based on the concept of Hogg et.al. (1975) are given by Büning (1999) and Büning and Kössler (1998).

Applying this idea now we can obtain a "double-adaptive test". On the one hand the type of the statistics (Jonckheere, Tryon-Hettmansperger, RM of Fairley and Fligner (1987) or possibly other) is chosen with respect to the sample sizes $n_1, \ldots, n_c$ and to the subset of the parameter space. On the other hand the choice of the score generating function (1) for the chosen type of rank test is based on some measures classifying the (unknown) underlying distribution function.

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