Propositional Logic: Review
Propositional logic

- **Logical constants**: true, false
- **Propositional symbols**: P, Q, S, ... (atomic sentences)
- Wrapping **parentheses**: ( … )
- Sentences are combined by **connectives**:
  - $\wedge$ ...and [conjunction]
  - $\lor$ ...or [disjunction]
  - $\Rightarrow$ ...implies [implication / conditional]
  - $\Leftrightarrow$ ..is equivalent [biconditional]
  - $\neg$ ...not [negation]
- **Literal**: atomic sentence or negated atomic sentence
Examples of PL sentences

• \((P \land Q) \rightarrow R\)
  “If it is hot and humid, then it is raining”

• \(Q \rightarrow P\)
  “If it is humid, then it is hot”

• \(Q\)
  “It is humid.”

• A better way:
  \(Ho = \text{“It is hot”}\)
  \(Hu = \text{“It is humid”}\)
  \(R = \text{“It is raining”}\)
Propositional logic (PL)

• A simple language useful for showing key ideas and definitions
• User defines a set of propositional symbols, like P and Q.
• User defines the **semantics** of each propositional symbol:
  – P means “It is hot”
  – Q means “It is humid”
  – R means “It is raining”
• A sentence (well formed formula) is defined as follows:
  – A symbol is a sentence
  – If S is a sentence, then \( \neg S \) is a sentence
  – If S is a sentence, then \( \sim S \) is a sentence
  – If S and T are sentences, then \( S \lor T \), \( S \land T \), \( S \rightarrow T \), and \( S \leftrightarrow T \) are sentences
  – A sentence results from a finite number of applications of the above rules
Some terms

• The meaning or **semantics** of a sentence determines its **interpretation**.

• Given the truth values of all symbols in a sentence, it can be “evaluated” to determine its **truth value** (True or False).

• A **model** for a KB is a “possible world” (assignment of truth values to propositional symbols) in which each sentence in the KB is True.
More terms

• A **valid sentence** or **tautology** is a sentence that is True under all interpretations, no matter what the world is actually like or what the semantics is. Example: “It’s raining or it’s not raining.”

• An **inconsistent sentence** or **contradiction** is a sentence that is False under all interpretations. The world is never like what it describes, as in “It’s raining and it’s not raining.”

• **P entails Q**, written $P \models Q$, means that whenever $P$ is True, so is $Q$. In other words, all models of $P$ are also models of $Q$. 
Truth tables

The five logical connectives:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>¬P</th>
<th>P ∧ Q</th>
<th>P ∨ Q</th>
<th>P → Q</th>
<th>P ↔ Q</th>
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</table>

A complex sentence:

<table>
<thead>
<tr>
<th>P</th>
<th>H</th>
<th>P ∨ H</th>
<th>(P ∨ H) ∧ ¬H</th>
<th>((P ∨ H) ∧ ¬H) → P</th>
</tr>
</thead>
<tbody>
<tr>
<td>False</td>
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Models of complex sentences

- $P \lor Q$
- $P \land Q$
- $P \Rightarrow Q$
- $P \Leftrightarrow Q$
Equivalences

Which one do you remember?
Inference rules

• **Logical inference** is used to create new sentences that logically follow from a given set of predicate calculus sentences (KB).

• An inference rule is **sound** if every sentence X produced by an inference rule operating on a KB logically follows from the KB. (That is, the inference rule does not create any contradictions)

• An inference rule is **complete** if it is able to produce every expression that logically follows from (is entailed by) the KB. (Note the analogy to complete search algorithms.)
Sound rules of inference

• Here are some examples of sound rules of inference
  – A rule is sound if its conclusion is true whenever the premise is true
• Each can be shown to be sound using a truth table

<table>
<thead>
<tr>
<th>RULE</th>
<th>PREMISE</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modus Ponens</td>
<td>A, A → B</td>
<td>B</td>
</tr>
<tr>
<td>And Introduction</td>
<td>A, B</td>
<td>A ∧ B</td>
</tr>
<tr>
<td>And Elimination</td>
<td>A ∧ B</td>
<td>A</td>
</tr>
<tr>
<td>Double Negation</td>
<td>¬¬A</td>
<td>A</td>
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<tr>
<td>Unit Resolution</td>
<td>A ∨ B, ¬B</td>
<td>A</td>
</tr>
<tr>
<td><strong>Resolution</strong></td>
<td><strong>A ∨ B, ¬B ∨ C</strong></td>
<td><strong>A ∨ C</strong></td>
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</table>
Proving things

• A proof is a sequence of sentences, where each sentence is either a premise or a sentence derived from earlier sentences in the proof by one of the rules of inference.
• The last sentence is the theorem (also called goal or query) that we want to prove.
• Example for the “weather problem” given above.

1 Hu Premise “It is humid”
2 Hu→Ho Premise “If it is humid, it is hot”
3 Ho Modus Ponens(1,2) “It is hot”
4 (Ho∧Hu)→R Premise “If it’s hot & humid, it’s raining”
5 Ho∧Hu And Introduction(1,3) “It is hot and humid”
6 R Modus Ponens(4,5) “It is raining”
Entailment and derivation

• **Entailment**: $KB \models Q$
  
  – $Q$ is entailed by $KB$ (a set of premises or assumptions) if and only if there is no logically possible world in which $Q$ is false while all the premises in $KB$ are true.
  
  – Or, stated positively, $Q$ is entailed by $KB$ if and only if the conclusion is true in every logically possible world in which all the premises in $KB$ are true.

• **Derivation**: $KB \vdash Q$
  
  – We can derive $Q$ from $KB$ if there is a proof consisting of a sequence of valid inference steps starting from the premises in $KB$ and resulting in $Q$
Two important properties for inference

**Soundness:** If \( KB \vdash Q \) then \( KB \models Q \)

- If \( Q \) is derived from a set of sentences \( KB \) using a given set of rules of inference, then \( Q \) is entailed by \( KB \).
- Hence, inference produces only real entailments, or any sentence that follows deductively from the premises is valid.

**Completeness:** If \( KB \models Q \) then \( KB \vdash Q \)

- If \( Q \) is entailed by a set of sentences \( KB \), then \( Q \) can be derived from \( KB \) using the rules of inference.
- Hence, inference produces all entailments, or all valid sentences can be proved from the premises.
Quick and dirty: Resolution

1) Transform set of formulae into conjunctive normal form
   =>  set of clauses
2) Add the negated goal/query
3) Apply resolution rule until either
   - no more application possible or
   - the empty clause is derived

If empty clause can be derived, then the goal follows from the set of formulae!
Propositional logic is a weak language

• Hard to identify “individuals” (e.g., Mary, 3)
• Can’t directly talk about properties of individuals or relations between individuals (e.g., “Bill is tall”)
• Generalizations, patterns, regularities can’t easily be represented (e.g., “all triangles have 3 sides”)
• First-Order Logic (abbreviated FOL or FOPC) is expressive enough to concisely represent this kind of information
  FOL adds relations, variables, and quantifiers, e.g.,
  • “Every elephant is gray”: \( \forall x \ (\text{elephant}(x) \rightarrow \text{gray}(x)) \)
  • “There is a white alligator”: \( \exists x \ (\text{alligator}(X) \land \text{white}(X)) \)
Propositional logic: Summary

• The process of deriving new sentences from old one is called **inference**.
  – **Sound** inference processes derives true conclusions given true premises
  – **Complete** inference processes derive all true conclusions from a set of premises

• A **valid sentence** is true in all worlds under all interpretations

• If an implication sentence can be shown to be valid, then—given its premise—its consequent can be derived

• Different logics make different **commitments** about what the world is made of and what kind of beliefs we can have regarding the facts
  – Logics are useful for the commitments they do not make because lack of commitment gives the knowledge base engineer more freedom

• **Propositional logic** commits only to the existence of facts that may or may not be the case in the world being represented
  – It has a simple syntax and simple semantics. It suffices to illustrate the process of inference
  – Propositional logic quickly becomes impractical, even for very small worlds