On Homomorphism Indistinguishability and Hypertree Depth

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What the next 20 minutes are about

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Preprint: arXiv:2404.10637
Slides: hu.berlin/icalp24

Counting Homomorphisms

Hypergraphs

GC — Guarded Logic with Counting Quantifiers

Results

From Tree Depth to (Strict) Hypertree Depth

So what?

Counting homomorphisms

$$\begin{aligned} \hom(F,G) &= \#\text{homs } F \to G \\ \mathrm{Hom}(\mathfrak{C},G) &= (\#\text{homs } F \to G)_{F \in \mathfrak{C}} \end{aligned}$$

Can be used to distinguish graphs. **Q**: How well does this work?

A: Very well. Many results that look like this:

Theorem

Let G, H be graphs and $\mathfrak{C}$ a class of graphs.

$$\mathrm{Hom}(\mathfrak{C},G) = \mathrm{Hom}(\mathfrak{C},H) \quad \Longleftrightarrow \quad G \equiv_X H$$

- X may be a logic, an algorithm, …
- Natural classes $\mathfrak{C}$ lead to natural X!

An incomplete history of homomorphism indistinguishability

G and H are homomorphism indistinguishable over

- all graphs iff isomorphic (Lovász 1967)
- Graphs of tree-width $\leq k$ iff indistinguishable by C^{k+1} (Dvořák 2010; Dell, Grohe, Rattan 2018)

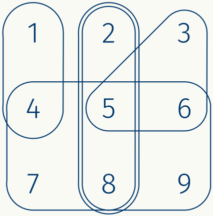
$\exists^{\geq n} x\ \varphi$

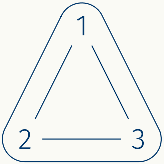
- Graphs of tree-depth $\leq m$ iff indistinguishable by C_m (Grohe 2020)
- and many more fancy results on *graphs*

Can we mirror that on hypergraphs?

What are Hypergraphs?

A hypergraph H is a finite set $V(H)$ of vertices and a finite **multi**set $E(H)$ of edges $e \subseteq V(H)$.





How to do logic on hypergraphs?

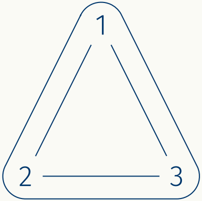
Incidence Graphs

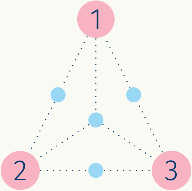
Definition

The incidence graph $I_H = (R, B, E)$ of H is defined as:

$$R = V(H) \qquad B = E(H) \qquad (e, v) \in E \iff v \in f_H(e)$$

i.e., draw an edge if v is contained in the hyperedge e.





Introducing GC — The Basics

- Blue and red Variables

$$\mathsf{VAR}_{\mathsf{e}} := \{\mathsf{e}_1, \ldots, \mathsf{e}_k\} \qquad \text{represent edges}$$

$$\mathsf{VAR}_{\mathsf{v}} := \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \ldots\} \qquad \text{represent vertices}$$

- Guard function $g : \mathbb{N}_{\geq 1} \rightharpoonup [k]$

$$\Delta_g := \bigwedge_{i \in \mathsf{dom}(g)} E(\mathsf{e}_{g(i)}, \mathsf{v}_i) \qquad \text{“}v_i \in e_{g(i)}\text{”}$$

$$\Delta_g := \top \text{ for } \mathsf{dom}(g) = \varnothing$$

- red vars are always guarded by some blue var

Introducing GC — Definition

We have...

Atomic formulas:

- $E(\mathsf{e}_i, \mathsf{v}_j)$, $\mathsf{v}_i = \mathsf{v}_j$, $\mathsf{e}_i = \mathsf{e}_j$

Negation, Conjunction and Disjunction:

- $\neg\varphi$, $(\varphi \land \psi)$, $(\varphi \lor \psi)$

Guarded Quantification:

- $\exists^{\geq n}(\mathsf{v}_{i_1}, \dots, \mathsf{v}_{i_\ell}).(\Delta_g \land \varphi)$
- $\exists^{\geq n}(\mathsf{e}_{i_1}, \dots, \mathsf{e}_{i_\ell}).(\Delta_g \land \varphi)$

Δ_g guards *all* free red variables in φ: $\mathrm{free}_{\mathsf{v}}(\varphi) \subseteq \mathrm{free}_{\mathsf{v}}(\Delta_g)$

GC – Example

Find φ that expresses:

> *All hyperedges containing at least 3 vertices are pairwise disjoint.*

$$\varphi := \neg\exists^{\geq 1}(\mathsf{e}_1, \mathsf{e}_2) \,.\, (\underbrace{\top}_{\text{Guard}} \wedge (\neg \mathsf{e}_1 = \mathsf{e}_2 \wedge \psi_{\text{n. disjoint}} \wedge \psi_{\geqslant 3 \text{ vert.}}))$$

with

$$\psi_{\text{n. disjoint}} := \exists^{\geq 1}(\mathsf{v}_1) \,.\, (\underbrace{E(\mathsf{e}_1, \mathsf{v}_1)}_{\text{Guard}} \wedge \bigwedge_{j \in \{1,2\}} E(\mathsf{e}_j, \mathsf{v}_1))$$

$$\psi_{\geqslant 3 \text{ vert.}} := \bigwedge_{j \in \{1,2\}} \exists^{\geq 3}(\mathsf{v}_1) \,.\, (\underbrace{E(\mathsf{e}_j, \mathsf{v}_1)}_{\text{Guard}} \wedge E(\mathsf{e}_j, \mathsf{v}_1))$$

Previous work

Theorem (Dvořák 2010; Dell, Grohe, Rattan 2018)

Let G, H be graphs, $k \in \mathbb{N}_{\geq 1}$.

$$\mathsf{Hom}(\mathsf{TW}_k, G) = \mathsf{Hom}(\mathsf{TW}_k, H) \quad \iff \quad G \equiv_{\mathsf{C}^{k+1}} H$$

Theorem (Scheidt, Schweikardt 2023)

Let G, H be hypergraphs, $k \in \mathbb{N}_{\geq 1}$.

$$\mathsf{Hom}(\mathsf{GHW}_k, G) = \mathsf{Hom}(\mathsf{GHW}_k, H) \quad \iff \quad G \equiv_{\mathsf{GC}^k} H$$

GHW_k are the hypergraphs of generalised hypertree width $\leq k$.

Now: Complete the picture

Theorem (Grohe 2020)

Let G, H be graphs, $k \in \mathbb{N}_{\geq 1}$.

$$\mathsf{Hom}(\mathsf{TD}_k, G) = \mathsf{Hom}(\mathsf{TD}_k, H) \quad \iff \quad G \equiv_{\mathsf{C}_k} H$$

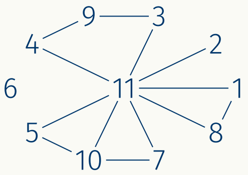
Theorem

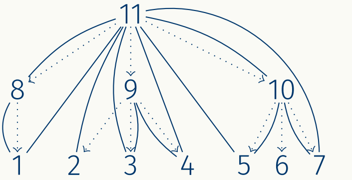
Let G, H be hypergraphs, $k \in \mathbb{N}_{\geq 1}$.

$$\mathsf{Hom}(\mathsf{SHD}_k, G) = \mathsf{Hom}(\mathsf{SHD}_k, H) \quad \iff \quad G \equiv_{\mathsf{GC}_k} H$$

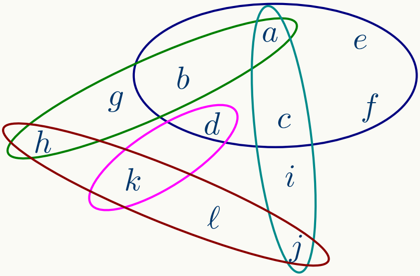
SHD_k are the hypergraphs of *strict* hypertree depth $\leq k$.

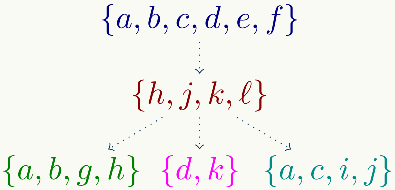
Elimination Forest “Connections only to ancestors”.





(Stolen from Grohe 2020)





Strict Elimination Forest

(Stolen from Adler et al.)

Comparing HD_k and SHD_k

Theorem

For all H: $\mathsf{shd}(H) \leq \mathsf{hd}(H) + 1$.

— But why bother?

Theorem

$$\begin{aligned} & \mathrm{Hom}(\mathrm{SHD}_k, G) = \mathrm{Hom}(\mathrm{SHD}_k, H) \\ \iff\ & \mathrm{Hom}(\mathrm{ISHD}_k, I_G) = \mathrm{Hom}(\mathrm{ISHD}_k, I_H) \end{aligned}$$

Theorem

There are G_k, H_k such that

$$\begin{aligned} \mathrm{Hom}(\mathrm{HD}_k, G_k) &= \mathrm{Hom}(\mathrm{HD}_k, H_k), \quad \textit{but} \\ \mathrm{Hom}(\mathrm{IHD}_k, I_{G_k}) &\neq \mathrm{Hom}(\mathrm{IHD}_k, I_{H_k}) \end{aligned}$$

Take away

Theorem

Let G, H be hypergraphs and $k \in \mathbb{N}_{\geq 1}$.

1. $\mathsf{Hom}(\mathsf{SHD}_k, G) = \mathsf{Hom}(\mathsf{SHD}_k, H) \iff G \equiv_{\mathsf{GC}_k} H$
2. $\mathsf{Hom}(\mathsf{GHW}_k, G) = \mathsf{Hom}(\mathsf{GHW}_k, H) \iff G \equiv_{\mathsf{GC}^k} H$

- Hom. Indistinguishability is interesting on hypergraphs.
- Hyperedges may hide complexity.
- GC fits hypergraph params like C fits graph params.

Further Reading

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