Counting Homomorphisms from Hypergraphs of Bounded Generalised Hypertree Width

A Logical Characterisation

Benjamin Scheidt   Nicole Schweikardt

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What is this about?

Slides at hu.berlin/highlights23
Have a look at our poster 👀
arXiv:2303.10980

1 Counting Homomorphisms 2 from Hypergraphs
3 of Bounded Generalised Hypertree Width:
4 A Logical Characterisation

Counting homomorphisms...

- $\text{hom}(F, G) =$ number of homomorphisms from F to G.
- $\text{Hom}(\mathfrak{C}, G) = \big(\text{hom}(F, G)\big)_{F \in \mathfrak{C}}$
- Use this to distinguish graphs

Many alternative characterisations of *homomorphism indistinguishability*:

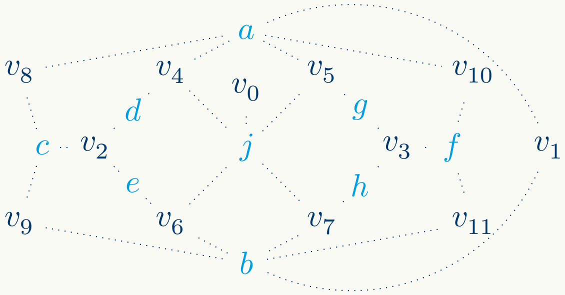
Theorem

Let G, H be graphs and $\mathfrak{C}$ a class of graphs.

$$\text{Hom}(\mathfrak{C}, G) = \text{Hom}(\mathfrak{C}, H) \quad \iff \quad G \equiv_X H$$

X may be a logic, an algorithm, ...

A hypergraph H is a finite set $V(H)$ of vertices and a finite set $E(H)$ of edges $e \subseteq V(H)$.



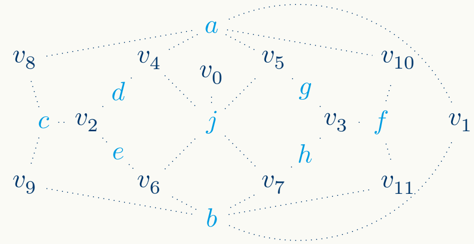
- Generalisation of treewidth to hypergraphs.
- Based on (Hyper)tree decompositions – as usual:
    - Bags of vertices associated with each node of the tree
    - F.a. vertices v: the bags containing v must form a connected subtree.

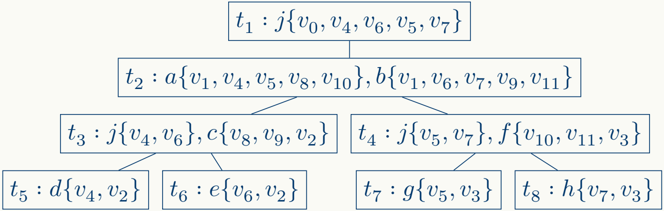
The twist:

- Every bag has to be *covered* by a set of hyperedges
    - I.e. the union of the hyperedges must be a superset of the bag.
- The width of a decomposition is the size of the biggest cover.

Example

F:


a
v8
v4
v0
v5
v10
d
g
c
v2
j
v3
f
v1
e
h
v9
v6
v7
v11
b


$t_1 : j\{v_0, v_4, v_6, v_5, v_7\}$
$t_2 : a\{v_1, v_4, v_5, v_8, v_{10}\}, b\{v_1, v_6, v_7, v_9, v_{11}\}$
$t_3 : j\{v_4, v_6\}, c\{v_8, v_9, v_2\}$
$t_4 : j\{v_5, v_7\}, f\{v_{10}, v_{11}, v_3\}$
$t_5 : d\{v_4, v_2\}$
$t_6 : e\{v_6, v_2\}$
$t_7 : g\{v_5, v_3\}$
$t_8 : h\{v_7, v_3\}$

A Logical Characterisation…

…of homomorphism indistinguishability over the class of hypergraphs of generalised hypertree width at most k.

Theorem

Let G, H be hypergraphs, $k \in \mathbb{N}_{\geqslant 1}$.

$$\mathsf{Hom}(\mathsf{GHW}_k, G) = \mathsf{Hom}(\mathsf{GHW}_k, H) \quad \Longleftrightarrow \quad G \equiv_{\mathsf{GC}^k} H$$

Theorem (Dvořák 2010; Dell, Grohe, and Rattan 2018)

Let G, H be graphs, $k \in \mathbb{N}_{\geqslant 1}$.

$$\mathsf{Hom}(\mathsf{TW}_k, G) = \mathsf{Hom}(\mathsf{TW}_k, H) \quad \Longleftrightarrow \quad G \equiv_{\mathsf{C}^{k+1}} H$$

...of homomorphism indistinguishability over the class of hypergraphs of generalised hypertree width at most k.

Theorem

Let G, H be hypergraphs, $k \in \mathbb{N}_{\geqslant 1}$.

$$\mathsf{Hom}(\mathsf{GHW}_k, G) = \mathsf{Hom}(\mathsf{GHW}_k, H) \iff G \equiv_{\mathsf{GC}^k} H$$

Proof.

Exercise 🙃

□

Introducing GC^k — The Basics

- Blue and red Variables

$$\begin{aligned} \mathsf{VAR} := \{\mathsf{e}_1, \dots, \mathsf{e}_k\} & \qquad \text{represent edges} \\ \cup \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \dots\} & \qquad \text{represent vertices} \end{aligned}$$

- Guard function $g : \mathbb{N}_{\geqslant 1} \rightharpoonup [k]$

$$\Delta_g := \bigwedge_{i \in \mathsf{dom}(g)} E(\mathsf{e}_{g(i)}, \mathsf{v}_i) \qquad \text{“}v_i \in e_{g(i)}\text{”}$$

$$\Delta_g := \top \text{ for } \mathsf{dom}(g) = \emptyset$$

- red variables are always guarded by some blue variable

Introducing GC^k — Definition

We have...

Atomic formulas:

- $E(\mathsf{e}_i, \mathsf{v}_j)$, $\mathsf{v}_i = \mathsf{v}_j$, $\mathsf{e}_i = \mathsf{e}_j$

Negation, Conjunction and Disjunction:

- $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$

Guarded Quantification:

- $\exists^{\geq n}(\mathsf{v}_{i_1}, \dots, \mathsf{v}_{i_\ell}).(\Delta_g \wedge \varphi)$
- $\exists^{\geq n}(\mathsf{e}_{i_1}, \dots, \mathsf{e}_{i_\ell}).(\Delta_g \wedge \varphi)$

where Δ_g has to guard *all* free red variables in φ.

Find φ that expresses:

All hyperedges containing at least 3 vertices are pair-wise disjoint.

$$\varphi := \neg\exists^{\geq 1}(\mathsf{e}_1, \mathsf{e}_2) \,.\, (\underbrace{\top}_{\text{Guard}} \wedge (\neg \mathsf{e}_1 = \mathsf{e}_2 \wedge \psi_{\text{not disjoint}} \wedge \psi_{\geqslant 3 \text{ vertices}}))$$

with

$$\psi_{\text{not disjoint}} := \exists^{\geq 1}(\mathsf{v}_1) \,.\, (\underbrace{E(\mathsf{e}_1, \mathsf{v}_1)}_{\text{Guard}} \wedge \bigwedge_{j \in \{1,2\}} E(\mathsf{e}_j, \mathsf{v}_1))$$

$$\psi_{\geqslant 3 \text{ vertices}} := \bigwedge_{j \in \{1,2\}} \exists^{\geq 3}(\mathsf{v}_1) \,.\, (\underbrace{E(\mathsf{e}_j, \mathsf{v}_1)}_{\text{Guard}} \wedge E(\mathsf{e}_j, \mathsf{v}_1))$$

Concluding Remarks

Theorem

The following statements are equivalent:

1. *Ex.* $\varphi \in \mathsf{GC}^k$ *s.t.* $G \models \varphi$ *and* $H \not\models \varphi$
2. *Ex.* F *with* $\mathsf{ghw}(F) \leq k$ *s.t.* $\mathsf{hom}(F, G) \neq \mathsf{hom}(F, H)$

So...

- ...how hard is it to compute the homomorphism vector?
- ...EF-games for GC^k?
- ...does Grohe 2020a generalise to hypertree-depth?
- ...Weisfeiler-Leman on hypergraphs? Ongoing work!

Related Work i

- 📝 Cai, Jin-Yi, Martin Fürer, and Neil Immerman (1992). “An optimal lower bound on the number of variables for graph identification”.
  DOI: 10.1007/BF01305232.
- 📝 Courcelle, Bruno (1993). “Graph Grammars, Monadic Second-Order Logic And The Theory Of Graph Minors”.
  DOI: 10.1090/conm/147.
- 📝 Dell, Holger, Martin Grohe, and Gaurav Rattan (2018). “Lovász Meets Weisfeiler and Leman”.
  DOI: 10.4230/LIPIcs.ICALP.2018.40.

Related Work ii

- 📝 Dvořák, Zdeněk (2010). “On recognizing graphs by numbers of homomorphisms”.
  DOI: 10.1002/jgt.20461.
- 📝 Gottlob, Georg, Nicola Leone, and Francesco Scarcello (2002). “Hypertree Decompositions and Tractable Queries”.
  DOI: 10.1006/jcss.2001.1809.
- 📝 Grohe, Martin (2020a). “Counting Bounded Tree Depth Homomorphisms”.
  DOI: 10.1145/3373718.3394739.

Related Work iii

- 📝 Grohe, Martin (2020b). “Word2vec, Node2vec, Graph2vec, X2vec: Towards a Theory of Vector Embeddings of Structured Data”. DOI: 10.1145/3375395.3387641.
- 📝 Grohe, Martin (2021). “The Logic of Graph Neural Networks”. DOI: 10.1109/LICS52264.2021.9470677.
- 📝 Mančinska, Laura and David E. Roberson (2020). “Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs”. DOI: 10.1109/FOCS46700.2020.00067.

On Hypertree Decompositions

Definition (Gottlob, Leone, and Scarcello 2002)

A complete generalised hypertree decomposition $D := (T, cover, bag)$ of H is a tree T with functions $cover : V(T) \to E(H)$ and $bag : V(T) \to V(H)$ s.t.

1. F.a. $e \in E(H)$ ex. $n \in V(T)$ s.t. $e \in cover(n)$ and $f(e) \subseteq bag(n)$
2. F.a. $v \in V(H)$ the subgraph T_v induced by $V_v := \{n \in V(T) \mid v \in bag(n)\}$ is a tree.
3. F.a. $n \in V(T)$ we have $bag(n) \subseteq \bigcup_{e \in cover(n)} f(e)$.

Definition

A complete entangled hypertree decomposition $D := (T, cover, bag)$ of H is a tree T with functions $cover : V(T) \to E(H)$ and $bag : V(T) \to V(H)$ s.t.

1. F.a. $e \in E(H)$ ex. $n \in V(T)$ s.t. $e \in cover(n)$ and $f(e) \subseteq bag(n)$
2. F.a. $v \in V(H)$ the subgraph T_v induced by $V_v := \{n \in V(T) \mid v \in bag(n)\}$ is a tree.
3. F.a. $n \in V(T)$ we have $bag(n) = \bigcup_{e \in cover(n)} f(e)$.
4. F.a. $e \in E(H)$ the subgraph T_e induced by $V_e := \{n \in V(T) \mid e \in cover(n)\}$ is a tree.