A tutorial on order-invariant logics

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- p is a graph property, if the following is true for all graphs G and H:

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- q is a k-ary graph-query, if the following is true:

  if $\pi : G \cong H$, then for all $a_1, \ldots, a_k \in V^G$,

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- I.e., graph-properties and queries are closed under isomorphisms.

Logics expressing graph-properties and queries

Classical logics like, e.g.

- FO (first-order logic: Boolean combinations + quantification over nodes)
- LFP (least fixed point logic: FO + inductive definitions of relations)

express graph-properties and queries in a straightforward way.

Example: The query

$$q(G) \;=\; \{\, x \in V^G \;:\; x \text{ lies on a triangle} \,\}$$

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Drawback:

FO and LFP are too weak to express (some) computationally easy properties, e.g., properties concerning the size of V^G or E^G.
Stronger logics like, e.g., SO or ESO can express computationally hard properties and queries.

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- Extend the expressive power of a logic by allowing formulas to also use arithmetic predicates like $<$, $+$, $\times$, ..., *Halt*, ... on V^G.
- For this, identify V^G with the set $[n] := \{0, 1, \ldots, n-1\}$ for $n = |V^G|$ and interpret $<$, $+$, $\times$, ..., *Halt*, ... in the natural way.

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Definition: Let $\mathcal{L}$ be a logic (e.g., FO, MSO, LFP).

A $\mathcal{L}(E, <)$-formula $\varphi(\vec{x})$ is order-invariant on $G = (V^G, E^G)$ $\iff$
for all tuples of nodes $\vec{a}$ in V^G, for all linear orders $<_1$ and $<_2$ on V^G,

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A $\mathcal{L}(E, <, +)$-formula $\varphi(\vec{x})$ is addition-invariant on $G = (V^G, E^G)$ $\iff$ for all tuples of nodes $\vec{a}$ in V^G, for all linear orders $<_1$ and $<_2$ on V^G, and the matching addition relations $+_1$, $+_2$,

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Definition: Let $\mathcal{L}$ be a logic (e.g., FO, MSO, LFP).

A $\mathcal{L}(E, <, +, \times)$-formula $\varphi(\vec{x})$ is $(+, \times)$-invariant on $G = (V^G, E^G)$ $\iff$ for all tuples of nodes $\vec{a}$ in V^G, for all linear orders $<_1$ and $<_2$ on V^G, and the matching addition relations $+_1, +_2$, and the according multiplications $\times_1, \times_2$,

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For Arb-invariant sentences, shortly write $G \models \varphi$ for $(G, <_1, +_1, \times_1 \dots) \models \varphi$.

Example

- An addition-invariant FO$(E, <, +)$-sentence φ such that

$$G \models \varphi \quad \Longleftrightarrow \quad |V^G| \text{ is odd.}$$

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Thus:

order-inv FO $<$ addition-inv FO $<$ $(+,\times)$-inv FO $<$ Arb-invariant FO.

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Exercise: Get rid of C!

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Order-invariance for unary signatures is decidable

Let $\{C_1, \ldots, C_\ell\}$ consist of unary relation symbols (i.e., node colors).

Theorem: Order-invariance of a given $\mathrm{FO}(C_1, \ldots, C_\ell, <)$-sentence φ (on the class of all finite colored sets) is decidable.

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  Definition: L is commutative iff
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- Commutativity of regular string-languages is decidable.

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- **Goal:** Construct, for each $n \in \mathbb{N}$, a FO($C, <, +$)-sentence $\tilde{\psi}_n$ such that $n \notin H \iff \tilde{\psi}_n$ is addition-invariant.

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Overview

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Represent a finite unordered binary tree T by a $\{E, D\}$-structure $\mathcal{A}_T$ where $(A, E^{\mathcal{A}}) = T$ and $D^{\mathcal{A}}$ is the transitive closure of $E^{\mathcal{A}}$ (i.e., the descendant-relation).

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Exercise: Work out the details for the general case!

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  - for any 3 nodes u, v, w such that
    $E(u, v)$ and $E(v, w)$ and $(D(w, x_h) \vee w = x_h)$ we have:
    v is a 1-child iff w is a 2-child.

$\square$

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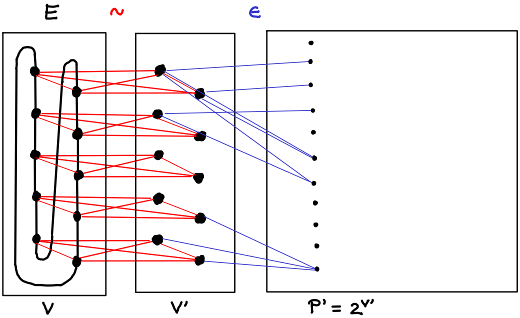
Let $\sigma = \{E, \sim, \in, V, V', P'\}$ with binary $E, \sim, \in$ and unary V, V', P'.

Represent a graph $G = (V, E)$ on $2n$ nodes by the σ-structure $\mathcal{S}_{2n}(G)$:



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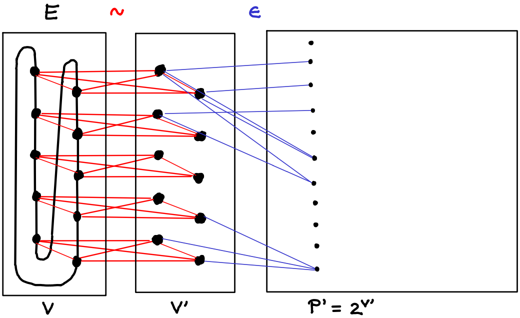

Theorem (Otto):

There is an order-inv. FO($\sigma, <$)-sentence φ_{conn}, but no FO(σ)-sentence, such that for all $n \in \mathbb{N}$ and all graphs G on $2n$ nodes we have

$$\mathcal{S}_{2n}(G) \models \varphi_{conn} \iff G \text{ is connected.}$$

Successor-invariant FO

By a much more elaborate construction, one can also show:

Theorem (Rossman, LICS'03)

On the class of all finite structures,
successor-invariant FO is strictly more expressive than FO.

FO+MOD$_2$ $<$ order-invariant FO+MOD$_2$

FO+MOD$_2$: the extension of FO by modulo 2 counting quantifiers

$\exists^{0 \bmod 2} x\ \psi(x)$: the number of nodes x satisfying $\psi(x)$ is congruent 0 modulo 2.

Theorem (Niemistö):

There is an order-invariant FO+MOD$_2(E)$-sentence $\varphi_{\textit{even cycles}}$ that is satisfied by a finite directed graph $G = (V^G, E^G)$ iff

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$\square$

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  - CMSO cannot (for showing this, use a variant of EF-games).

Overview

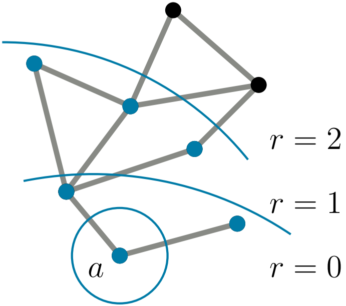
Neighborhoods

Graph $G = (V^G, E^G)$

Distance *dist*(u, v) : length of shortest path from u to v in undirected version of G.

Shell $S_r(a)$ of nodes at distance exactly r from a.

Ball $N_r(a)$ of radius r at a in G.



Neighborhoods

Graph $G = (V^G, E^G)$

Distance $dist(u, v)$: length of shortest path from u to v in undirected version of G.

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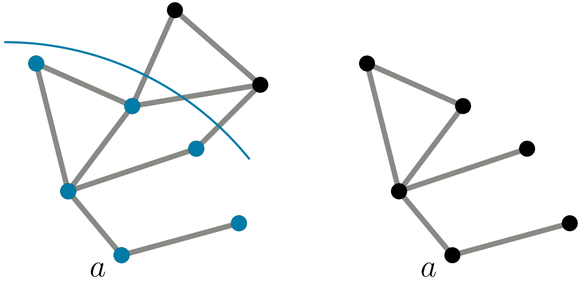
Ball $N_r(a)$ of radius r at a in G.

Neighborhood $\mathcal{N}_r^G(a)$ of radius r at a in G : induced subgraph of G on $N_r(a)$.



Gaifman-local queries

- For a list $a = a_1, \ldots, a_k$ of nodes, $N_r(a) = N_r(a_1) \cup \cdots \cup N_r(a_k)$.
- The *r*-neighborhood $\mathcal{N}_r^G(a)$ is the induced subgraph of G on $N_r(a)$.

Definition: Let q be a k-ary graph query. Let $f : \mathbb{N} \to \mathbb{N}$.
q is called $f(n)$-local if there is an n_0 such that for every $n \geqslant n_0$ and every graph G with $|V^G| = n$, the following is true for all k-tuples a and b of nodes:

$$\text{if} \quad (\mathcal{N}_{f(n)}^G(a), a) \cong (\mathcal{N}_{f(n)}^G(b), b) \quad \text{then} \quad a \in q(G) \iff b \in q(G).$$

Gaifman-locality of FO

Theorem:

- For every graph query q that is FO-definable,
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- For every graph query q that is FO-definable on graphs with arbitrary numerical predicates (i.e., q is definable in Arb-invariant FO),
  there is a constant c such that q is $(\log n)^c$-local.
  (Anderson, van Melkebeek, S., Segoufin '11)

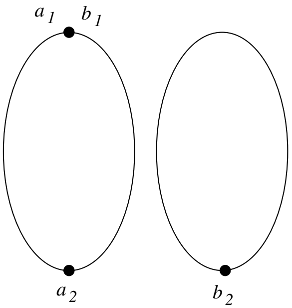
Use locality for proving non-expressibility

Example: The reachability query

$$\text{REACH}(G) := \{ (a_1, a_2) : \text{there is a directed path from } a_1 \text{ to } a_2 \text{ in } G \}$$

is not $\frac{n}{5}$-local an thus cannot be expressed in Arb-invariant FO.

Proof: Consider the graph G:



Use locality for proving non-expressibility

Similarly, one obtains that the following queries are not definable in Arb-invariant FO:

- Does node x lie on a cycle?
- Does node x belong to a connected component that is acyclic?
- Is node x reachable from a node that belongs to a triangle?
- Do nodes x and y have the same distance to node z?

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- Assume that q is *not* $(\log n)^c$-local (for any $c \in \mathbb{N}$), and modify $\mathcal{C}$ to obtain a circuit family computing

$$\text{PARITY} := \{w \in \{0,1\}^* : |w|_1 \text{ is even}\}.$$

- This contradicts known lower bounds in circuit complexity theory (Håstad'86).

Proof of Gaifman-locality theorem (2/5)

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- *Known:* A unary graph query $q(x)$ is definable in Arb-invariant FO $\iff$ it is computed by an AC^0-circuit family of constant depth and polynomial size.
  (implicit in Immerman'87)

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Theorem: *(Håstad '86)*

There exist $\ell, m_0 > 0$ such that for all $m \geqslant m_0$, no circuit of depth d and size $2^{\ell \cdot m^{1/(d-1)}}$ computes parity on m bits.

Contradiction for $c = 2d$, since $2^{\ell \cdot m^{1/(d-1)}} > 2^{\ell\cdot(\log n)^2} = n^{\ell \log n} > p(n)$. $\square$

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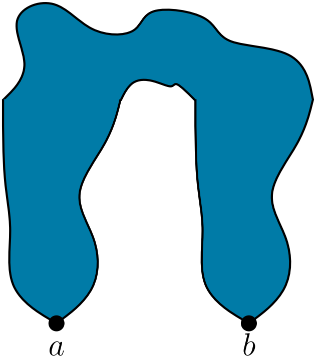

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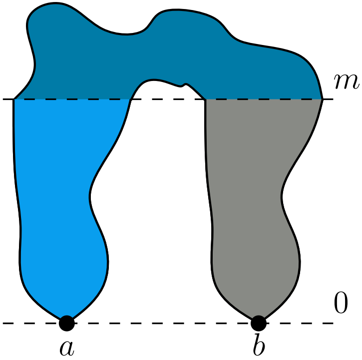

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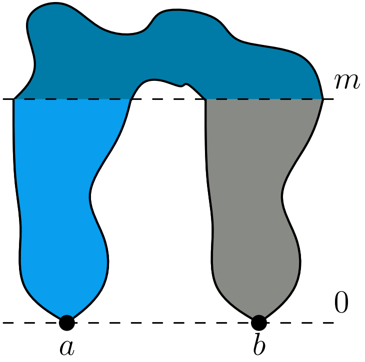

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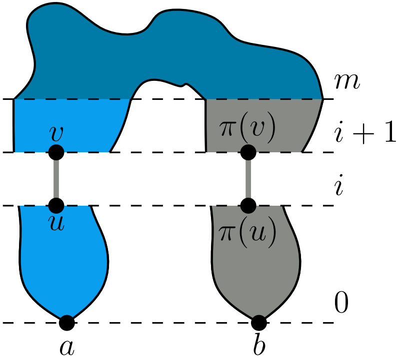

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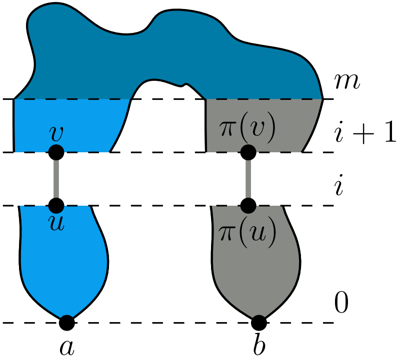

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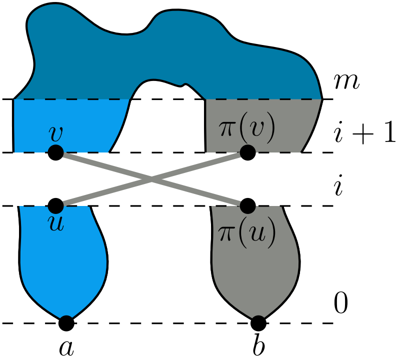

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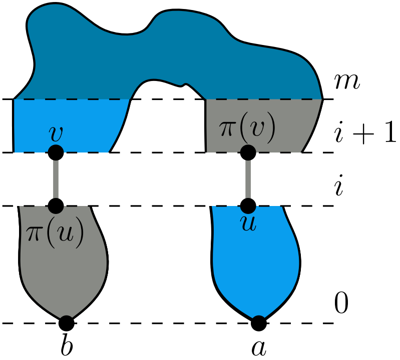

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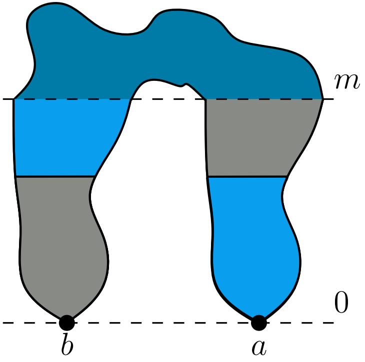

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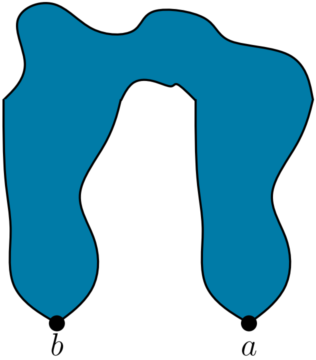

Proof of Gaifman-locality theorem (4/5)

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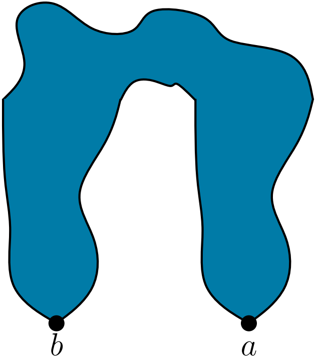

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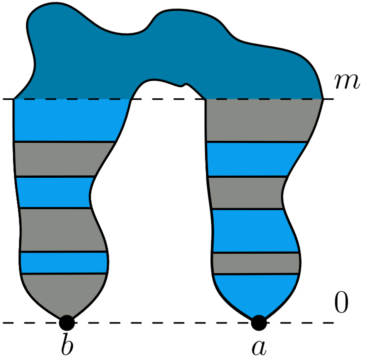

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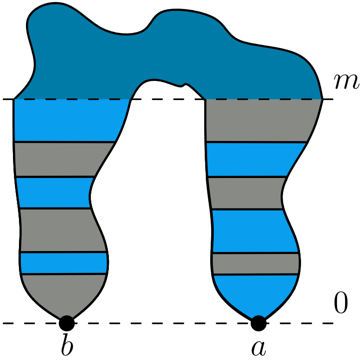

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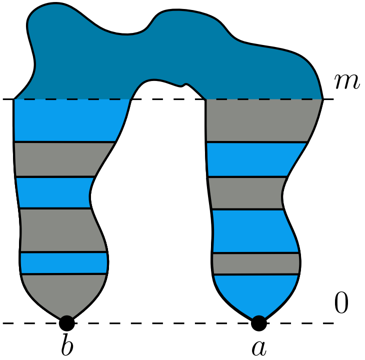

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- This yields a circuit $\tilde{C}$ of the same size and depth as C which, on input $w \in \{0,1\}^m$ does the same as C on input (G_w, a).
  Thus, $\tilde{C}$ accepts iff $|w|_1$ is even. □

Summary: Gaifman-locality of Arb-invariant FO (1/2)

Theorem: *(Anderson, Melkebeek, S., Segoufin '11)*

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(Show that in graphs satisfying (1), reachability by paths of length $(\log n)^{d+1}$ can be expressed in $(+, \times)$-invariant FO)

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Summary: Gaifman-locality of Arb-invariant FO (2/2)

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For every $c \in \mathbb{N}$ there is a $\mathrm{FO}(<, +, \times, S)$-formula $bij_c(x, y)$ such that for all $n \in \mathbb{N}$, all $S \subseteq [n] := \{0, \ldots, n{-}1\}$, all $a, i < n$ we have

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- Iterate this for $c{+}1$ times to express that there is a path of length $\ell(n)^{c+1} \geqslant (\log n)^c$ from x to y. □

Locality of Arb-invariant FO+MOD$_p$

In a similar way, we can also prove:

Theorem: *(Harwath, S., '13)*

Let p be a prime power and let $k \in \mathbb{N}$ be coprime with p.

For every k-ary query q expressible in Arb-invariant FO+MOD$_p$*, there is a $c \in \mathbb{N}$ such that q is $(\log n)^c$-shift-local w.r.t. k.*

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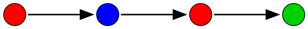
Corollary: Easy proof that reachability is not definable in Arb-invariant FO+MOD$_p$, for prime power p.

Overview

Represent words as labeled graphs

- (labeled) chain-graphs . . . . .this chain-graph represents the string $rbrg$.



Edges correspond to the successor relation "*succ*" on the positions of the string.
Write $<$ -inv-FO(*succ*) for order-invariant FO on these graphs.

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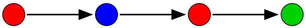

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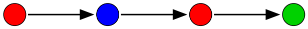

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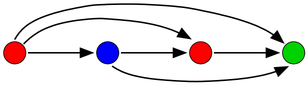

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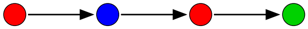

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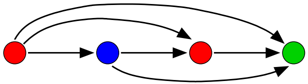

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Write $+$-inv-FO($<$) for addition-invariant FO on these graphs.
Note that on these graphs, $<$ -inv-FO($<$) is the same as FO($<$).

Known results

FO(*succ*)	=	locally threshold testable languages	(Thomas)
FO($<$)	=	star-free regular languages	(McNaughton, Papert)
MSO($<$)	=	regular languages	(Büchi, Elgot, Trakhtenbrot)

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The following problem is undecidable: (Benedikt, Segoufin, 2005)

ORDER-INVARIANCE ON FINITE LABELED CHAIN-GRAPHS:
Input: a $\mathrm{FO}(<, E, C_1, \ldots, C_\ell)$-sentence φ
Question: Is φ order-invariant on all finite labeled chain-graphs?

Known results

$\mathrm{FO}(succ)$ = locally threshold testable languages (Thomas)

$\mathrm{FO}(<)$ = star-free regular languages (McNaughton, Papert)

$\mathrm{MSO}(<)$ = regular languages (Büchi, Elgot, Trakhtenbrot)

$\mathrm{MSO}(<) = \mathrm{MSO}(succ) = \mathrm{MLFP}(<) = \mathrm{MLFP}(succ)$

$<$-inv-$\mathrm{FO}(succ) = \mathrm{FO}(succ)$ (Benedikt, Segoufin, 2005)

$+$-inv-$\mathrm{MLFP}(succ) \supseteq$ DLIN (S., 2004)

$+$-inv-$\mathrm{MSO}(succ)$ = the linear time hierarchy (More & Olive 1997, S. 2004)

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  - L is closed under swaps iff for all strings u, v, e, x, y, z such that e, f are idempotents (i.e., for all u, v we have $uev \in L$ iff $ue^2v \in L$), we have

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$$u\,e\,x\,f\,y\,e\,z\,f\,v \in L \quad \iff \quad u\,e\,z\,f\,y\,e\,x\,f\,v \in L.$$

- Show that every string-language definable in $<$-inv-FO($succ$) is aperiodic and closed under swaps.

  (For this, you can use Ehrenfeucht-Fraïssé games.) □

Different situation for FO+MOD_2

Proposition (Harwath, S., '13): $<$-inv-FO+MOD_2(*succ*) $\neq$ FO+MOD_2(*succ*)
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FO+MOD$_2$ $<$ order-invariant FO+MOD$_2$

FO+MOD$_2$: the extension of FO by modulo 2 counting quantifiers

$\exists^{0 \bmod 2} x\, \psi(x)$: the number of nodes x satisfying $\psi(x)$ is congruent 0 modulo 2.

Theorem (Niemistö):

There is an order-invariant FO+MOD$_2(E)$-sentence $\varphi_{\textit{even cycles}}$ that is satisfied by a finite directed graph $G = (V^G, E^G)$ iff

(1) G is a disjoint union of directed cycles, and

(2) the number of even-length cycles is even.

Proof:

- (1) can be expressed in FO: "every node has in- and out-degree 1"
- Every G satisfying (1) is the cycle decomposition of a permutation π.
- G has an even number of even-length cycles $\iff$
  π is an even permutation, i.e., $\operatorname{sgn}(\pi) = 1$ $\iff$
  π has an even number of inversions (i, j) such that $i < j$ and $\pi(i) > \pi(j)$.

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Situation:

- $w \in 1^*\, 2\, 0^*\, 1^*\, 0^* \quad\Longrightarrow\quad$ 2 cycles, sum of lenghts: $|w|_1 + 1$
- $w \in 1^*\, 0^*\, 1^*\, 2\; 0^* \quad\Longrightarrow\quad$ 1 cycle, length: $|w|_1 + 1$

$\square$

Further results proved by the algebraic approach

Theorem:

- A tree-language is definable in $<$-invariant FO(*succ*) iff it is definable in FO(*succ*). (Benedikt, Segoufin '09)
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Let L be a regular language. The following are equivalent:

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L regular $\implies \exists\, r \in \mathbb{N} : \forall\, x \in \Sigma^*$, x^r is idempotent

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Observation:

Given an automaton for a regular language L, it is decidable whether L is closed under transfers.

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Proof: (2)$\Longleftrightarrow$(3): easy.

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Proof: (2)$\iff$(3): easy.

(2)$\implies$(1): easy. E.g.: $|w| \equiv 1 \bmod 2 \quad\iff$

$$w \models \exists x\, \exists z\, \big(x + x = z \wedge \forall y\,(\,y < z \vee y = z\,)\big)$$

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- $Z_i^q \;=\; \{w : |w| \equiv i \bmod q\}$.

(4) *L is closed under transfers.*

Proof:

(2)$\Longleftrightarrow$(3): easy.

(2)$\Longrightarrow$(1): easy. E.g.: $|w| \equiv 1 \bmod 2 \iff$

$$w \models \exists x\, \exists z\, \big(x + x = z \,\wedge\, \forall y\, (\, y < z \vee y = z\,)\big)$$

(1)$\Longrightarrow$(4): use Ehrenfeucht-Fraïssé games.

Regular languages definable in $+$-inv-FO($<$)

Theorem: *(S., Segoufin, 2010)*

Let L be a regular language. The following are equivalent:

(1) *L is definable in $+$-inv-FO($<$).*

(2) *L is definable in $\mathrm{FO}_{\mathrm{Card}}(<)$.*

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(2)$\Longleftrightarrow$(3): easy.

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$$w \models \exists x\, \exists z\, \big(x + x = z \,\wedge\, \forall y\, (\, y < z \vee y = z\,)\big)$$

(1)$\Longrightarrow$(4): use Ehrenfeucht-Fraïssé games.

(4)$\Longrightarrow$(2): use tools from algebraic automata theory.

Proof of (4)$\Longrightarrow$(2): L is regular & closed under transfers.
Goal: Show that L is definable in $\mathrm{FO}_{\mathrm{Card}}(<)$.

Choose a suitable number $q > 0$.

skip proof

For $0 \leqslant i < q$ let $L_i \ := \ L \cap Z_i^q$, where $Z_i^q \ := \ \{w \in \Sigma^* : |w| \equiv i \bmod q\}$.

Clearly, $L = \bigcup_{0 \leqslant i < q} L_i$.

Proof of (4)$\Longrightarrow$(2): L is regular & closed under transfers.

Goal: Show that L is definable in $\mathrm{FO}_{\mathrm{Card}}(<)$.

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For $0 \leqslant i < q$ let $L_i \ := \ L \cap Z_i^q$, where $Z_i^q \ := \ \{w \in \Sigma^* : |w| \equiv i \bmod q\}$.

Clearly, $L = \bigcup_{0 \leqslant i < q} L_i$.

Goal: Show that L_i is definable in $\mathrm{FO}_{\mathrm{Card}}(<)$.

Proof of (4)$\Longrightarrow$(2): L is regular & closed under transfers.
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For $0 \leqslant i < q$ let $L_i := L \cap Z_i^q$, where $Z_i^q := \{w \in \Sigma^* : |w| \equiv i \bmod q\}$.

Clearly, $L = \bigcup_{0 \leqslant i < q} L_i$.

Goal: Show that L_i is definable in $\text{FO}_{\text{Card}}(<)$.

Approach: Find a regular language M_i such that

- $L_i = M_i \cap Z_i^q$,
- The minimal DFA for M_i does not contain any counter.

Then, apply

Theorem: *(McNaughton & Papert, 1971)*

Let M be a regular language. Then, the following are equivalent:

(1) The minimal DFA for M does not contain any counter.

(2) M is definable in FO($<$) *(i.e., M is star-free regular).*

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).
Goal: Show that L is closed under transfers.

skip proof

For contradiction, assume that L is not closed under transfers. Then:

$$\exists\, x, y, z, u, v \in \Sigma^* \;:\quad |x| = |z| \quad \text{and}$$
$$u\, x^r\, x\, y\, z^r\, v \;\in\; L \quad \text{and} \quad u\, x^r\, y\, z\, z^r\, v \;\notin\; L.$$

Thus:

$$\forall\, \alpha,\ \beta \geqslant 1 \;:\quad \Big(\, u\, x^{\alpha r}\, x\, y\, z^{\beta r}\, v \;\in\; L \quad \text{and} \quad u\, x^{\alpha r}\, y\, z\, z^{\beta r}\, v \;\notin\; L \,\Big).$$

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).

Goal: Show that L is closed under transfers. **Proof by contradiction:**

Situation: Fixed $x, y, z, u, v \in \Sigma^*$ with $|x| = |z|$ such that

$$\forall\, \alpha,\ \beta \geqslant 1 \ : \quad \big(\, u\, x^{\alpha r}\, x\, y\, z^{\beta r}\, v \ \in\ L \quad \text{and} \quad u\, x^{\alpha r}\, y\, z\, z^{\beta r}\, v\ \notin\ L\,\big).$$

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Idea: Consider the languages

$$L_1 := \{\ w \in\ uyv\,x\,(xz \mid zz)^* \ :\ |w|_x, |w|_z \geqslant r,\ \ |w|_x \equiv 1\ [r],\ \ |w|_z \equiv 0\ [r]\ \},$$

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Lemma 1: If L is definable in $+$-inv-FO($<$), then there is a FO($<, +$)-formula that separates L_1 from L_2.

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).

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Proof: Construct a FO($<,+$) interpretation that, on $w \in uyvx\,(xz\,|\,zz)^*$, evaluates φ on the corrensponding string w' of the form $u\,(x)^*\,y\,(z)^*\,v$.

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).

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Situation: Fixed $x, y, z, u, v \in \Sigma^*$ with $|x| = |z|$ such that

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Clearly,
- $w \in L_1 \implies w' \in L \implies w' \models \varphi$,
- $w \in L_2 \implies w' \notin L \implies w' \not\models \varphi$. □

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).

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Lemma 1: If L is definable in $+$-inv-FO($<$), then there is a FO($<, +$)-formula that separates L_1 from L_2.

Lemma 2: No formula of FO($<, +$) can separate L_1 from L_2.

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).

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Proof idea: Use Ehrenfeucht-Fraïssé games.

Proof of (1)$\Longrightarrow$(4): Let L be regular and definable by a φ in $+$-inv-FO($<$).

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CONTRADICTION. $\square$

Regular languages definable in $+$-inv-FO($<$)

Theorem: *(S., Segoufin, 2010)*

Let L be a regular language. The following are equivalent:

(1) *L is definable in $+$-inv-FO($<$).*

(2) *L is definable in $\mathrm{FO}_{\mathrm{Card}}(<)$.*

(3) *L is a finite union of languages of the form $S \cap Z_i^q$, where*

- *S is star-free regular (i.e., S is FO($<$)-definable)*
- $Z_i^q = \{w : |w| \equiv i \bmod q\}$.

(4) *L is closed under transfers.*

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(4) *L is closed under transfers.*

Question: What happens if the linear order $<$ on the string is not available?

We first consider the case where only the successor relation "*succ*" is available.

Regular languages definable in $+$-inv-FO($succ$)

Theorem: *(S., Segoufin, 2010)*

Let L be a regular language. The following are equivalent:

(1) *L is definable in $+$-inv-FO($succ$).*

(2) *L is definable in $\mathrm{FO}_{\mathrm{Card}}(succ)$.*

(3) *L is a finite union of languages of the form $T \cap Z_i^q$, where*

- *T is locally threshold testable (i.e., T is FO($succ$)-definable)*
- $Z_i^q \;=\; \{w : |w| \equiv i \bmod q\}$.

(4) *L is closed under transfers and under swaps.*

Proof method: Similar as for the previous theorem.

Definition: L is closed under swaps $\iff$

for all $e, f, x, y, z \in \Sigma^*$ such that e, f are idempotent we have

$$e\,x\,f\,y\,e\,z\,f \;\;=_L\;\; e\,z\,f\,y\,e\,x\,f$$

Observation: Given an automaton for a regular language L, it is decidable whether L is closed under transfers and under swaps.

Regular languages definable in $+$-inv-FO($succ$)

Theorem: *(S., Segoufin, 2010)*

Let L be a regular language. The following are equivalent:

(1) *L is definable in $+$-inv-FO($succ$).*

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- *T is locally threshold testable (i.e., T is FO($succ$)-definable)*
- $Z_i^q \ = \ \{w : |w| \equiv i \bmod q\}$.

(4) *L is closed under transfers and under swaps.*

By combining this with the poly-logarithmic-locality of Arb-invariant FO, we obtain:

Theorem: *(Anderson, Melkebeek, S., Segoufin, 2011)*

Let L be a regular language. Then,
L is definable in Arb-invariant FO($succ$) iff L is definable in $\mathrm{FO}_{\mathrm{Card}}(succ)$.

Regular languages definable in $+$-inv-FO(*succ*)

Theorem: *(S., Segoufin, 2010)*

Let L be a regular language. The following are equivalent:

(1) *L is definable in $+$-inv-FO(succ).*

(2) *L is definable in $\text{FO}_{\text{Card}}(succ)$.*

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- *T is locally threshold testable (i.e., T is FO(succ)-definable)*
- $Z_i^q \ = \ \{w : |w| \equiv i \bmod q\}$.

(4) *L is closed under transfers and under swaps.*

The result extends from words to trees:

Theorem: *(Harwath, S., 2012)*

Let L be a regular tree language. The following are equivalent:

(1) *L is definable in $+$-inv-FO(S_1, S_2).*

(2) *L is definable in $\text{FO}_{\text{Card}}(S_1, S_2)$.*

(3) *L is closed under transfers and swaps.*

Regular languages definable in $+$-inv-FO$(=)$

Theorem: *(S., Segoufin, 2010)*

Let L be a regular language. The following are equivalent:

(1) *L is definable in $+$-inv-FO$(=)$.*

(2) *L is definable in $\mathrm{FO}_{\mathrm{Card}}(=)$.*

(3) *L is commutative, closed under transfers and under swaps.*

Definition: L is commutative $\iff$

$$\forall\, m \in \mathbb{N} \quad \forall\, a_1, \ldots, a_m \in \Sigma \quad \forall \text{ permutations } \pi \text{ of } \{1, \ldots, m\} \;:$$

$$a_1 a_2 \cdots a_m \;\in\; L \quad \iff \quad a_{\pi(1)} a_{\pi(2)} \cdots a_{\pi(m)} \;\in\; L.$$

An open question

Open Question:

Are all languages definable in addition-invariant FO regular?

Known: (S., Segoufin, 2010)

- Arb-invariant FO can define non-regular languages, e.g., $L = \{ w \in \{1\}^* : |w| \text{ is a prime number } \}$.
- Every deterministic context-free language definable in addition-invariant FO is regular.
- Every commutative language definable in addition-invariant FO is regular.
- Every bounded language definable in addition-invariant FO is regular.

Bounded languages

Definition: (Ginsburg & Spanier, 1964)

$L \subseteq \Sigma^*$ is bounded $\iff$

$$\exists\, k \in \mathbb{N} \text{ and } k \text{ strings } w_1, \ldots, w_k \in \Sigma^* \text{ such that } L \subseteq w_1^* w_2^* \cdots w_k^* .$$

Theorem: *(S., Segoufin, 2010)*

Every bounded language definable in $+$-*inv*-FO($<$) *is regular.*

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Theorem: *(S., Segoufin, 2010)*

Every bounded language definable in $+$-inv-FO($<$) is regular.

Proof method:

- Identify $w_1^* w_2^* \cdots w_k^*$ with $\mathbb{N}^k$ via $(x_1, \ldots, x_k) \in \mathbb{N}^k \;\widehat{=}\; w_1^{x_1} w_2^{x_2} \cdots w_k^{x_k}$.
  Thus: $L \subseteq w_1^* w_2^* \cdots w_k^* \;\widehat{=}\; S(L) \subseteq \mathbb{N}^k$.
- Note that $S(L)$ is semi-linear, since L is definable in $+$-inv-FO($<$).
- Reason about semi-linear sets ...

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- Note that $S(L)$ is semi-linear, since L is definable in $+$-inv-FO$(<)$.
- Reason about semi-linear sets ...

Corollary:

Every commutative language definable in $+$-inv-FO$(<)$ is regular.

Characterization of colored sets definable in $+$-inv-FO

Definition: A colored finite set is a finite relational structure over a finite signature that contains only unary relation symbols.

> *Theorem:* *(S., Segoufin, 2010)*
>
> *Over the class of colored finite sets, $+$-inv-FO$(=)$ and $\mathrm{FO}_{\mathrm{Card}}(=)$ have the same expressive power.*

Proof:

- Every $+$-inv-FO$(=)$ sentence over colored sets defines a commutative language.
- Every commutative language definable in $+$-inv-FO$(<)$ is regular.
- Every regular language definable in $+$-inv-FO$(=)$ is definable in $\mathrm{FO}_{\mathrm{Card}}(=)$.

$\square$

Characterization of colored sets definable in $+$-inv-FO

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Theorem: *(S., Segoufin, 2010)*

Over the class of colored finite sets, $+$-inv-FO$(=)$ and $\mathrm{FO}_{\mathrm{Card}}(=)$ have the same expressive power.

Note: $\mathrm{FO}_{\mathrm{Card}}(=)$ is a logic (with a decidable syntax); $+$-inv-FO$(=)$ is not.

More precisely: The following problem is undecidable:

Input: a FO$(<, +, C)$-sentence φ (C a unary relation symbol)

Question: Is φ addition-invariant on all finite $\{C\}$-structures ?

Overview

Gaifman-locality

$$\text{If } \big(\mathcal{N}_r^G(a), a\big) \cong \big(\mathcal{N}_r^G(b), b\big) \text{ then } \big(a \in q(G) \iff b \in q(G)\big).$$

Known:

- Queries definable in order-invariant FO are Gaifman-local with respect to a constant locality radius. (Grohe, Schwentick '98)
- Queries definable in Arb-invariant FO are Gaifman-local with respect to a poly-logarithmic locality radius. (Anderson, Melkebeek, S., Segoufin '11)

Open Question:

- How about addition-invariant FO:
  is it Gaifman-local with respect to a **constant** locality radius?

Hanf-locality

A graph property p is Hanf-local w.r.t. locality radius r, if any two graphs having the same r-neighbourhood types with the same multiplicities, are not distinguished by p.

Known:

- Properties of graphs definable in FO are Hanf-local w.r.t. a constant locality radius. (Fagin, Stockmeyer, Vardi '95)
- Properties of strings or trees definable by order-invariant FO are Hanf-local w.r.t. a constant locality radius. (Benedikt, Segoufin '09)
- Properties of strings definable by Arb-invariant FO are Hanf-local w.r.t. a poly-logarithmic locality radius. (Anderson, van Melkebeek, S., Segoufin '11)
- Properties of strings definable by Arb-invariant FO$+$MODp, for **odd** prime powers p, are Hanf-local w.r.t. a poly-logarithmic locality radius. For **even** p, they aren't. (Harwath, S. '13)

Open Question:

- Which of these results generalise from strings to arbitrary finite graphs?

Decidable Characterisations

Open Question:

Are there decidable characterisations of

- order-invariant FO?
- addition-invariant FO?
- $(+, \times)$-invariant FO?

Known:

- On finite strings and trees: order-invariant FO $\equiv$ FO. (Benedikt, Segoufin '10)
- On finite coloured sets: addition-invariant FO $\equiv$ FO enriched by "cardinality modulo" quantifiers. (S., Segoufin '10)

Thank You!