Quantitative Analysis of Time Petri Nets Used for Modelling Biochemical Networks

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Outline

Definitions
- Petri Net
- Time Petri Net

Main Property
- State Space Reduction

Applications
- Reachability Graph
- T-Invariants
- Time Paths in unbounded TPNs
- Time Paths in bounded TPNs
- Time PN and Timed PN

Conclusion
chemical reactions -> atomic actions -> Petri net transitions

\[2 \text{ NAD}^+ + 2 \text{ H}_2\text{O} \rightarrow 2 \text{ NADH} + 2 \text{ H}^+ + \text{O}_2\]
r1: A -> B
\( r1: A \rightarrow B \)
\( r2: B \rightarrow C + D \)
\( r3: B \rightarrow D + E \)

\[ \rightarrow \text{alternative reactions} \]
r1: A -> B
r2: B -> C + D
r3: B -> D + E
r4: F -> B + a

r6: C + b -> G + c
r7: D + b -> H + c

-> concurrent reactions
r1: A -> B
r2: B -> C + D
r3: B -> D + E
r4: F -> B + a
r5: E + H <-> F
r6: C + b -> G + c
r7: D + b -> H + c
r8: H <-> G

-> reversible reactions
r1: A -> B
r2: B -> C + D
r3: B -> D + E
r4: F -> B + a
r5: E + H <-> F
r6: C + b -> G + c
r7: D + b -> H + c
r8: H <-> G

-> reversible reactions
- hierarchical nodes
r1: $A \rightarrow B$

r2: $B \rightarrow C + D$

r3: $B \rightarrow D + E$

r4: $F \rightarrow B + a$

r5: $E + H \leftrightarrow F$

r6: $C + b \rightarrow G + c$

r7: $D + b \rightarrow H + c$

r8: $H \leftrightarrow G$

r9: $G + b \rightarrow K + c + d$

r10: $H + 28a + 29c \rightarrow 29b$

r11: $d \rightarrow 2a$
r1: A -> B
r2: B -> C + D
r3: B -> D + E
r4: F -> B + a
r5: E + H <-> F
r6: C + b -> G + c
r7: D + b -> H + c
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r8: H <-> G
r9: G + b -> K + c + d
r10: H + 28a + 29c -> 29b
r11: d -> 2a
-> properties as time-less net

INA
ORD HOM NBM PUR CSV SCF CON SC Ft0 tF0 Fp0 pF0 MG SM FC EFC ES
N Y N Y N Y Y N X Y N N Y N Y Y
CPI CTI B SB REV DSt BSt DTr DCF L LV L&S
N Y N N Y N ? N Y Y Y N

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September 2004
TRANSFORMATION, Ex1

T-INVARIANT

-> properties as time-less net

INA
ORD HOM NBM PUR CSV SCF CON SC Ft0 tF0 Fp0 pF0 MG SM FC EFC ES
N Y N Y N Y Y N X Y N N Y N Y Y
CPI CTI B SB REV DSt BSt DTr DCF L LV L&S
N Y N N Y N ? N Y Y Y N
TRANSFORMATION, Ex1

T-INVARIANTE

-> properties as time net

INA
ORD HOM NBM PUR CSV SCF CON SC Ft0 tF0 Fp0 pF0 MG SM FC EFC ES
N Y N Y N Y N Y N N Y N Y Y
CPI CTI B SB REV DSt BSt DTr DCF L LV L&S
N Y Y N N N ? N Y Y Y N
TRANSFORMATION, Ex2

-> properties as time-less net

INA
ORD HOM NBM PUR CSV SCF CON SC Ft0 tF0 Fp0 pF0 MG SM FC EFC ES
N Y N Y N Y N Y Y N N Y Y Y
CPI CTI B SB REV DSt BSt DTr DCF L LV L&S
N Y N N Y N ? N N Y Y N

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TRANSFORMATION, Ex2

T-INVARIANTE1
T-INVARIANTE2

-> properties as time-less net

INA
ORD HOM NBM PUR CSV SCF CON SC Ft0 tF0 Fp0 pF0 MG SM FC EFC ES
N Y N Y N Y Y N Y N N Y Y Y
CPI CTI B SB REV DSt BSt DTr DCF L LV L&S
N Y N N Y N ? N N Y Y N

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**TRANSFORMATION, Ex2**

- prod_A
- A
- r1
- r2
- B
- C
- cons_B
- cons_C

`->` properties as **time net**

<table>
<thead>
<tr>
<th>Property</th>
<th>INA</th>
<th>ORD</th>
<th>HOM</th>
<th>NBM</th>
<th>PUR</th>
<th>CSV</th>
<th>SCF</th>
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CPI, CTI, B, SB, REV, DST, BST, DTR, DCF, L, LV, L&S

- N
- Y
- Y
- N
- N
- N
- ?
- N
- Y
- Y
- Y
- Y
- N
Definition (Petri Net)

The structure $N = (P, T, F, V, m_0)$ is a Petri Net (PN), iff
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- $P$—set of places
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- $F \subseteq (P \times T) \cup (T \times P)$ und $dom(F) \cup cod(F) = P \cup T$
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- \( V : F \rightarrow \mathbb{N}^+ \) (weights of edges)
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- \( V : F \rightarrow \mathbb{N}^+ \) (weights of edges)

- \( m_0 : P \rightarrow \mathbb{N} \) (initial marking)
**Example**

The image shows a Petri net with places and transitions labeled as follows:

- **Places:**
  - $P_1$
  - $P_2$
  - $P_3$

- **Transitions:**
  - $t_1$
  - $t_2$
  - $t_3$
  - $t_4$

- **Arcs:**
  - A directed arc from $P_1$ to $P_2$ labeled with 2.

The diagram illustrates the flow and dependencies between places and transitions in a Petri net model.
Example

\[ m_0 = (0, 1, 1) \]
Example

Petri Net

- $m_0 = (0, 1, 1)$
- $t_1^- = (0, 1, 0)$
Example

- $m_0 = (0, 1, 1)$
- $t_1^- = (0, 1, 0)$
- $t_1^+ = (1, 0, 0)$
**Petri Net**

**Example**

- \( m_0 = (0, 1, 1) \)
- \( t_1^- = (0, 1, 0) \) \( t_1^+ = (1, 0, 0) \)
- \( \Delta(t_1) = -t_1^- + t_1^+ = (1, -1, 0) \)
Firing transition

Definition

- A transition \( t \in T \) is **enabled (may fire)** at a marking \( m \) iff all input-places of \( t \) have enough tokens.
Firing transition

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- A transition $t \in T$ is enabled (may fire) at a marking $m$ iff all input-places of $t$ have enough tokens e.g. $t^- \leq m$. 
Firing transition

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- A transition $t \in T$ is **enabled (may fire)** at a marking $m$ iff all input-places of $t$ have enough tokens e.g. $t^- \leq m$.
- When an enabled transition $t$ at a marking $m$ fires, a **successor** marking $m'$ is reached.
Firing transition

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  denoted by $m \xrightarrow{t} m'$.
Example

\[
N_1: \quad \begin{array}{c}
\text{t}_1 & \text{t}_2 \\
P_2 & P_1 \\
\text{t}_3 & \text{t}_4
\end{array}
\]
firing transition

Example

Petri Net
Time Petri Net

Definitions
Main Property
Applications
Conclusion

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firing transition

Example

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Example

Petri Net

Time Petri Net
firing transition

Example

[Diagram of a Petri Net with transitions labeled t₁, t₂, t₃, and t₄, and places labeled P₁, P₂, and P₃]
firing transition

Example
firing transition

Example

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Quantitative Analysis of TPNs

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Time Petri Net

Definition (Time Petri net)

The structure $Z = (P, T, F, V, m_0, I)$ is called a **Time Petri net (TPN)** iff:
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- $S(Z) := (P, T, F, V, m_0)$ is a PN (skeleton of $Z$)
- $I : T \rightarrow \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\})$ and
Definition (Time Petri net)

The structure \( Z = (P, T, F, V, m_o, I) \) is called a **Time Petri net (TPN)** iff:

- \( S(Z) := (P, T, F, V, m_o) \) is a PN (skeleton of \( Z \))
- \( I : T \rightarrow \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\}) \) and \( l_1(t) \leq l_2(t) \) for each \( t \in T \), where \( I(t) = (l_1(t), l_2(t)) \).
Definition (FTPN)

A TPN is called finite Time Petri net (FTPN) iff

\[ I : T \rightarrow \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \.]
Time Petri Net

Example

\[ Z_1: \]

[Image of a Petri Net diagram]

\[ t_1 \quad t_2 \quad t_3 \quad t_4 \]

\[ p_1 \quad p_2 \quad p_3 \]

\[ [1,5] \quad [0,3] \quad [2,4] \quad [2,3] \]
Example

$\mathbf{m}_0 = (0, 1, 1)$ $p$-marking
Time Petri Net

Example

- $m_0 = (0, 1, 1)$  $p$-marking
- $h_0 = (0, \#, \#, 0)$  $t$-marking
Definition (state)

Let $Z = (P, T, F, V, m_0, I)$ be a TPN and $h : T \rightarrow \mathbb{R}_0^+ \cup \{\#\}$. $z = (m, h)$ is called a **state** in $Z$ iff:

- $m$ is a $p$-marking in $Z$, e.g. $m$ is a marking in $S(Z)$.
- $h$ is a $t$-marking in $Z$, e.g. $\forall t ( (t \in T \land t - \leq m) \rightarrow h(t) \in \mathbb{R}_0^+ \land h(t) \leq lft(t))$, and $\forall t ( (t \in T \land t - \not\leq m) \rightarrow h(t) = \#)$. 
Definition (state)

Let $Z = (P, T, F, V, m_o, l)$ be a TPN and $h : T \rightarrow \mathbb{R}_0^+ \cup \{\#\}$. $z = (m, h)$ is called a state in $Z$ iff:

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  \[
  \forall t \left( (t \in T \land t^- \leq m) \rightarrow (h(t) \in \mathbb{R}_0^+ \land h(t) \leq lft(t)) \right),
  \]
**Definition (state)**

Let $Z = (P, T, F, V, m_o, l)$ be a TPN and $h : T \rightarrow \mathbb{R}_0^+ \cup \{\#\}$. $z = (m, h)$ is called a **state** in $Z$ iff:

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  \]
  and
  \[
  \forall t \ ( (t \in T \land t^{-} \not\leq m) \quad \rightarrow \quad h(t) = \#).
  \]
Definition (state changing)
Definition (state changing)

Let \( Z = (P, T, F, V, m_o, l) \) be a TPN, \( \hat{t} \) be a transition in \( T \) and \( z = (m, h) \), \( z' = (m', h') \) be two states.

Then (a) the transition \( \hat{t} \) is ready to fire in the state \( z = (m, h) \), denoted by \( z \xrightarrow{\hat{t}} z' \), iff (i) \( \hat{t} \leq m \) and (ii) \( eft(\hat{t}) \leq h(\hat{t}) \).
Definition (state changing)

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Definition (state changing)

Let $Z = (P, T, F, V, m_0, l)$ be a TPN, $\hat{t}$ be a transition in $T$ and $z = (m, h)$, $z' = (m', h')$ be two states. Then

(a) the transition $\hat{t}$ is **ready** to fire in the state $z = (m, h)$, denoted by $z \xrightarrow{\hat{t}} z'$, iff

(i) $\hat{t}^- \leq m$ and
(ii) $eft(\hat{t}) \leq h(\hat{t})$. 

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Definition (state changing)

(b) the state $z = (m, h)$ is **changed** into the state $z' = (m', h')$ by firing the transition $\hat{t}$, denoted by $z \xrightarrow{\hat{t}} z'$, iff

(i) $t$ is ready to fire in the state $z = (m, h)$

(ii) $m' = m + \Delta \hat{t}$ and

(iii) $\forall t (t \in T \rightarrow h'(t) = \begin{cases} \# & \text{iff } t - \hat{t} \leq m' \\ h(t) & \text{iff } t - \hat{t} > m \land t - \hat{t} \leq m' \land F_t \cap F_{\hat{t}} = \emptyset \\ 0 & \text{otherwise} \end{cases})$. 

---

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state changing

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\[ \forall t \ (t \in T_{\rightarrow} h'(t) = \begin{cases} \# & \text{iff} \ t - \hat{t} \leq m' \\ h(t) & \text{iff} \ t - \hat{t} \leq m \land t - \hat{t} \leq m' \land F_t \cap F_{\hat{t}} = \emptyset \\ 0 & \text{otherwise} \end{cases} \]
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(iii) \( \forall t ( t \in T \rightarrow h'(t) = \begin{cases} \# & \text{iff } t^- \not\subseteq m' \\ h(t) & \text{iff } t^- \leq m \land t^- \leq m' \land Ft \cap F\hat{t} = \emptyset \\ 0 & \text{otherwise} \end{cases} ) \).
(c) the state \( z = (m, h) \) is **changed** into the state \( z' = (m', h') \) by the time elapsing \( \tau \in \mathbb{R}_0^+ \), denoted by \( z \xrightarrow{\tau} z' \), iff
state changing

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(i) \( m' = m \) and
state changing

Definition (state changing)

(c) the state $z = (m, h)$ is **changed** into the state $z' = (m', h')$ by the time elapsing $\tau \in \mathbb{R}_0^+$, denoted by $z \xrightarrow{\tau} z'$, iff

(i) $m' = m$ and
(ii) $\forall t \ (t \in T \land h(t) \neq \# \rightarrow h(t) + \tau \leq lft(t))$ i.e. the time elapsing $\tau$ is possible, and
(c) the state \( z = (m, h) \) is **changed** into the state \( z' = (m', h') \) by the time elapsing \( \tau \in \mathbb{R}_0^+ \), denoted by \( z \xrightarrow{\tau} z' \), iff

(i) \( m' = m \) and

(ii) \( \forall t \in T \wedge h(t) \neq \# \rightarrow h(t) + \tau \leq lft(t) \) i.e. the time elapsing \( \tau \) is possible, and

(iii) \( \forall t \in T \rightarrow h'(t) := \begin{cases} h(t) + \tau & \text{iff} \quad t^- \leq m' \\ \# & \text{iff} \quad t^- \not\leq m' \end{cases} \).
Time Petri Net

Example

\[
(m_0, \begin{pmatrix}
0 \\
\# \\
0
\end{pmatrix})
\]
Time Petri Net

Example

\[
\begin{pmatrix} 0 \\ \# \\ 0 \end{pmatrix} \stackrel{1.3}{\rightarrow} \begin{pmatrix} 1.3 \\ \# \\ 1.3 \end{pmatrix}
\]

\[
(m_0, \begin{pmatrix} 0 \\ \# \\ 0 \end{pmatrix}) \stackrel{1.3}{\rightarrow} (m_1, \begin{pmatrix} 1.3 \\ \# \\ 1.3 \end{pmatrix})
\]
Example

\[ z_0 \xrightarrow{1.3} (m_1, \begin{pmatrix} 1.3 \\ \# \\ \# \\ 1.3 \end{pmatrix}) \xrightarrow{1.0} (m_2, \begin{pmatrix} 2.3 \\ \# \\ \# \\ 2.3 \end{pmatrix}) \]
Example

\[
\begin{align*}
Z_1: & & P_1 & & P_2 & & P_3 \\
\{1,5\} & & t_1 & & t_2 & & t_3 & & t_4 \\
\end{align*}
\]

\[
Z_0 \xrightarrow{1.3} \xrightarrow{1.0} (m_2, \begin{pmatrix} 2.3 \\ \# \\ \# \\ 2.3 \end{pmatrix}) \xrightarrow{t_4}
\]
Time Petri Net

Example

\[ Z_0 \xrightarrow{1.3} Z_1 : (m_2, \begin{pmatrix} \# \\ \# \\ 2.3 \end{pmatrix}) \xrightarrow{t_4} (m_3, \begin{pmatrix} 2.3 \\ \# \\ 0.0 \end{pmatrix}) \]
Time Petri Net

Example

\[ Z_0 \xrightarrow{1.3} t_1 \xrightarrow{1.0} t_2 \xrightarrow{t_4} (m_3, \begin{pmatrix} 2.3 \\ \# \\ 0.0 \\ \# \end{pmatrix}) \xrightarrow{2.0} (m_4, \begin{pmatrix} 4.3 \\ \# \\ 2.0 \\ \# \end{pmatrix}) \]
Example

\[ Z_0 \xrightarrow{1.3} t_1 \xrightarrow{1.0} t_4 \xrightarrow{2.0} (m_4, \begin{pmatrix} 4.3 \\ \# \\ 2.0 \end{pmatrix}) \xrightarrow{t_1} \]
Time Petri Net

Example

\[
Z_0 \xrightarrow{1.3} \xrightarrow{1.0} t_4 \xrightarrow{2.0} (m_4, \begin{pmatrix} 4.3 \\ \# \\ 2.0 \end{pmatrix}) \xrightarrow{t_1} (m_5, \begin{pmatrix} \# \\ 0.0 \\ 2.0 \end{pmatrix})
\]
Example

\[
Z_0 \xrightarrow{1.3} Z_1 \xrightarrow{1.0} t_4 \xrightarrow{2.0} t_1 \xrightarrow{t_2} (m_5, \begin{pmatrix} 0.0 \\ 2.0 \\ \# \end{pmatrix})
\]
Example

\( Z_0 \xrightarrow{1.3} 1.0 \xrightarrow{t_4} 2.0 \xrightarrow{t_1} (m_5, \begin{pmatrix} \# \\ 0.0 \\ 2.0 \end{pmatrix}) \xrightarrow{t_2} (m_6, \begin{pmatrix} 0.0 \\ \# \\ \# \end{pmatrix}) \)
Transition sequences, Runs

Definition

- **transition sequence**: $\sigma = (t_1, \cdots, t_n)$
Transition sequences, Runs

**Definition**

- **transition sequence:**  $\sigma = (t_1, \cdots, t_n)$
- **run:**  $\sigma(\tau) = (t_1, \tau_1, \cdots, \tau_{n-1}, t_n)$
Transition sequences, Runs

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- **transition sequence:** $\sigma = (t_1, \cdots, t_n)$
- **run:** $\sigma(\tau) = (t_1, \tau_1, \cdots, \tau_{n-1}, t_n)$
- **feasible run:** $z_0 \xrightarrow{\tau_1} z_0^* \xrightarrow{t_1} z_1 \xrightarrow{\tau_2} \cdots \xrightarrow{t_n} z_n$
Transition sequences, Runs

**Definition**

- **transition sequence:** \( \sigma = (t_1, \cdots, t_n) \)
- **run:** \( \sigma(\tau) = (t_1, \tau_1, \cdots, \tau_{n-1}, t_n) \)
- **feasible run:** \[ Z_0 \xrightarrow{\tau_1} Z_0^* \xrightarrow{t_1} Z_1 \xrightarrow{\tau_2} \cdots \xrightarrow{t_n} Z_n \]
- **feasible transition sequence:** \( \sigma \) is feasible if there ex. a feasible run \( \sigma(\tau) \)
Reachable state, Reachable marking, State space

Definition

- $z$ is **reachable state** in $Z$ if there exists a feasible run $\sigma(\tau)$ and
  
  $z_0 \xrightarrow{\sigma(\tau)} z$
Reachable state, Reachable marking, State space

**Definition**

- \( z \) is **reachable state** in \( Z \) if there ex. a feasible run \( \sigma(\tau) \) and \( z_0 \xrightarrow{\sigma(\tau)} z \)
- \( m \) is **reachable marking** in \( Z \) if there ex. a reachable state \( z \) in \( Z \) with \( z = (m, h) \)
Reachable state, Reachable marking, State space

**Definition**

- $z$ is **reachable state** in $Z$ if there ex. a feasible run $\sigma(\tau)$ and $z_0 \xrightarrow{\sigma(\tau)} z$

- $m$ is **reachable marking** in $Z$ if there ex. a reachable state $z$ in $Z$ with $z = (m, h)$

- The set of all reachable states in $Z$ is the **state space** of $Z$ (denoted: $StSp(Z)$).
State class

Definition (state class)

Let $Z$ be a TPN and $\sigma$ be a feasible transition sequence. The set $C_{\sigma}$ is called a state class, iff
State class

Definition (state class)

Let $Z$ be a TPN and $\sigma$ be a feasible transition sequence. The set $C_{\sigma}$ is called a state class, iff 

\[ C_e := \{ z \mid \exists \tau (\tau \in \mathbb{R}_0^+ \land z_0 \xrightarrow{\tau} z) \} \]

Basis: $C_e := \{ z \mid \exists \tau (\tau \in \mathbb{R}_0^+ \land z_0 \xrightarrow{\tau} z) \}$
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Let $Z$ be a TPN and $\sigma$ be a feasible transition sequence. The set $C_\sigma$ is called a state class, iff

**Basis:**

$$C_e := \{ z \mid \exists \tau ( \tau \in \mathbb{R}_0^+ \land z_0 \xrightarrow{\tau} z) \}$$

**Step:** Let $C_\sigma$ be already defined. Then $C_{\sigma t}$ is derived from $C_\sigma$ by firing $t$ ($C_\sigma \xrightarrow{t} C_{\sigma t}$), iff

$$C_{\sigma t} := \{ z \mid \exists z_1 \exists z_2 \exists \tau ( z_1 \in C_\sigma \land \tau \in \mathbb{R}_0^+ \land z_1 \xrightarrow{t} z_2 \xrightarrow{\tau} z) \}.$$
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**Obviously:** $StSp(Z) = \bigcup_{\sigma} C_\sigma$
Properties

▶ static properties:

▶ dynamic properties:
Properties

- **static properties: being**
  - pure
  - ordinary
  - free choice
  - extended simple
  - conservative, etc.

- **dynamic properties:**
Properties

- **static properties: being**
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  - ordinary
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- **dynamic properties: being/having**
  - bounded
  - live
  - reachable marking/state
  - place- or transitions invariants
  - deadlocks, etc.
Properties

- static properties: being
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decidable **without knowledge** of the state space!

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  decidable **without knowledge** of the state space!

- dynamic properties: being/having
  - bounded
  - live
  - reachable marking/state
  - place- or transitions invariants
  - deadlocks, etc.

  decidable, if at all (TPN is equiv. to TM!),

  **with implicit/explicit knowledge** of the state space
Parametric Description of the State Space

Let $Z = [P, T, F, V, m_0, I]$ be a TPN and $\sigma = (t_1, \ldots, t_n)$ be a transition sequence in $Z$. 
$\delta(\sigma) = [m_\sigma, \Sigma_\sigma, B_\sigma]$ is the parametric description of $\sigma$, if
Parametric Description of the State Space

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$\quad \triangleright \ m_0 \xrightarrow{\sigma} m_{\sigma}$
Let $Z = [P, T, F, V, m_0, I]$ be a TPN and $\sigma = (t_1, \cdots, t_n)$ be a transition sequence in $Z$.\[\delta(\sigma) = [m_\sigma, \Sigma_\sigma, B_\sigma] \text{ is the parametric description of } \sigma, \text{ if}\]

$\begin{align*}
\triangleright & \quad m_0 \xrightarrow{\sigma} m_\sigma \\
\triangleright & \quad \Sigma_\sigma(t) \text{ is a term (in a FO Logic), } "1/2-interpreted" \text{ as a sum of variables for each transition } t
\end{align*}$
Let $Z = [P, T, F, V, m_0, I]$ be a TPN and $\sigma = (t_1, \cdots, t_n)$ be a transition sequence in $Z$. 

$\delta(\sigma) = [m_{\sigma}, \Sigma_{\sigma}, B_{\sigma}]$ is the parametric description of $\sigma$, if

- $m_0 \xrightarrow{\sigma} m_{\sigma}$
- $\Sigma_{\sigma}(t)$ is a term (in a FO Logic), "1/2–interpreted” as a sum of variables for each transition $t$
- $B_{\sigma}$ is a set of formulae (in a FO Logic), "1/2–interpreted” as a system of inequalities.
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Obviously $\delta(\sigma) = C_\sigma$. 

Louchka Popova-Zeugmann
Quantitative Analysis of TPNs
Example

\[ \sigma = (e) \Rightarrow \delta(\sigma) = C \]

\[ \sigma, (x, \#, \#, x) \in \Sigma \]

\[ 0 \leq x \leq 3 \]

\[ Louchka Popova-Zeugmann \]

Quantitative Analysis of TPNs
Example

\[ \sigma = (e) \quad \Rightarrow \quad \delta(\sigma) = C_e = \left\{ \left( \begin{array}{c} 0, 1, 1 \\ m_{\sigma} \end{array} \right), \left( x_1, \#, \#, x_1 \right) \right\} \mid 0 \leq x_1 \leq 3 \]
Example

\[ \sigma = (e) = \Rightarrow \delta(\sigma) = C e = \{ ((0, 1, 1), \ldots, (x_1, \#, \#, x_1)) | 0 \leq x_1 \leq 3 \} \]
Example

\[ \sigma = (e) \Rightarrow \delta(\sigma) = C e = \{ ((0,1,1), \Sigma) \mid 0 \leq x_1 \leq 3 \} \]

Louchka Popova-Zeugmann
Quantitative Analysis of TPNs
Example

\[
\sigma = (e) \Rightarrow \delta(\sigma) = C e = \{ (0,1,1), (0,1,2), (0,1,3) \} \cup \{ (x_1, \#, \#, x_1) \} | 0 \leq x_1 \leq 3 \}
\]
Example

\[ \sigma = (t_4, t_3) \]
Example

\[ \sigma = (t_4, t_3) \implies \delta(\sigma) = C_{t_4 t_3} = \]

\[ \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 + x_2 + x_3 \\ \# \\ \# \\ x_3 \end{pmatrix} \right\} \mid \begin{align*}
2 & \leq x_1 \leq 3, \quad x_1 + x_2 \leq 5 \\
2 & \leq x_2 \leq 4, \quad x_1 + x_2 + x_3 \leq 5 \\
0 & \leq x_3 \leq 3
\end{align*} \]
State Space Reduction

Theorem (1)

Let $Z$ be a TPN and $\sigma = (t_1, \cdots, t_n)$ be a feasible transition sequence in $Z$, with a run $\sigma(\tau)$ as an execution of $\sigma$, i.e.

$$z_0 \xrightarrow{\tau_0} t_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau_n} t_n \xrightarrow{\tau} z_n = (m_n, h_n),$$

and all $\tau_i \in \mathbb{R}_0^+$. Then, there exists a further feasible run $\sigma(\tau^*)$ of $\sigma$ with

$$z_0 \xrightarrow{\tau^*_0} t_0 \xrightarrow{\tau^*} \cdots \xrightarrow{\tau^*_n} t_n \xrightarrow{\tau^*} z_n^* = (m_n^*, h_n^*),$$

such that
Theorem (1 – continuation)

\[ z_0 \xrightarrow{\tau_0} t_0 \xrightarrow{\tau} t_n \xrightarrow{\tau} z_n = (m_n, h_n), \quad \tau_i \in \mathbb{R}_0^+. \]

\[ z_0 \xrightarrow{\tau_0^*} t_0 \xrightarrow{\tau_n^*} t_n \xrightarrow{\tau} z_n^* = (m_n^*, h_n^*) \]
Theorem (1 – continuation)

\[ z_0 \xrightarrow{\tau_0} t_0 \xrightarrow{\tau_n} t_n \rightarrow z_n = (m_n, h_n), \tau_i \in \mathbb{R}_0^+. \]

\[ z_0 \xrightarrow{\tau_0^*} t_0 \xrightarrow{\tau_n^*} t_n \rightarrow z_n^* = (m_n^*, h_n^*) \]

1. For each \( i, 0 \leq i \leq n \) holds: \( \tau_i^* \in \mathbb{N} \).
State Space Reduction

Theorem (1 – continuation)

\[ z_0 \xrightarrow{\tau_0} t_0 \xrightarrow{\tau_n} t_n \xrightarrow{\tau_n} z_n = (m_n, h_n), \tau_i \in \mathbb{R}^+_0. \]

\[ z_0 \xrightarrow{\tau_0^*} t_0 \xrightarrow{\tau_n^*} t_n \xrightarrow{\tau_n^*} z_n^* = (m_n^*, h_n^*), \tau_i^* \in \mathbb{N}. \]

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1. For each \( i, 0 \leq i \leq n \) holds: \( \tau_i^* \in \mathbb{N}. \)
2. For each enabled transition \( t \) at marking \( m_n(=m_n^*) \) it holds:
   2.1 \( h_n(t)^* = \lfloor h_n(t) \rfloor. \)
   2.2 \( \sum_{i=1}^{n} \tau_i^* = \lfloor \sum_{i=1}^{n} \tau_i \rfloor \)
Theorem (2 – similar to 1)

Let $Z$ be a TPN and $\sigma = (t_1, \ldots, t_n)$ be a feasible transition sequence in $Z$, with a run $\sigma(\tau)$ as an execution of $\sigma$, i.e.

$$z_0 \xrightarrow{\tau_0} t_0 \xrightarrow{} \cdots \xrightarrow{} t_n \xrightarrow{} z_n = (m_n, h_n),$$

and all $\tau_i \in \mathbb{R}_0^+$. Then, there exists a further feasible run $\sigma(\tau^*)$ of $\sigma$ with

$$z_0 \xrightarrow{\tau_0^*} t_0 \xrightarrow{} \cdots \xrightarrow{} t_n \xrightarrow{} z_n^* = (m_n^*, h_n^*).$$

such that
**State Space Reduction**

**Theorem (2 – continuation)**

1. For each $i, 0 \leq i \leq n$ the time $\tau_i^*$ is a natural number.
2. For each enabled transition $t$ at marking $m_n(= m_n^*)$ it holds:
   2.1 $h_n(t)^* = \lceil h_n(t) \rceil$.
   2.2 $\sum_{i=1}^{n} \tau_i^* = \lceil \sum_{i=1}^{n} \tau_i \rceil$
Definitions
Main Property
Applications
Conclusion

State Space Reduction

Example

\[ \sigma(\tau) := z_0 \rightarrow t_1 \rightarrow 0 \rightarrow t_2 \rightarrow 0 \rightarrow t_4 \rightarrow 1 \rightarrow t_5 \rightarrow 1 \rightarrow \ldots \]
State Space Reduction

Example

\[ \sigma = (t_1 t_3 t_4 t_2 t_3) \]

\[ \sigma(\tau) := z_0 \xrightarrow{0.7} t_1 \xrightarrow{0.0} t_3 \xrightarrow{0.4} t_4 \xrightarrow{1.2} t_2 \xrightarrow{0.5} t_3 \xrightarrow{1.4} z \]
State Space Reduction

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\[ \sigma = (t_1 t_3 t_4 t_2 t_3) \]

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State Space Reduction

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\[ \sigma \sigma(\tau) := Z_0 \xrightarrow{0.7} t_1 \xrightarrow{0.0} t_2 \xrightarrow{0.4} t_4 \xrightarrow{1.2} t_2 \xrightarrow{0.5} t_3 \xrightarrow{1.4} Z \]

\[ \sigma = (t_1 t_3 t_4 t_2 t_3) \]

\[ Z_2: \]

- \( t_4 \) from \([0,2] \)
- \( t_1 \) from \([0,2] \)
- \( t_2 \) from \([0,2] \)
- \( t_6 \) from \([0,2] \)
- \( t_3 \) from \([0,2] \)

- \( P_1 \)
- \( P_2 \)
- \( P_3 \)
- \( P_4 \)
- \( P_5 \)
Example

\[ \sigma = (t_1 t_3 t_4 t_2 t_3) \]

\[ \sigma(\tau) := Z_0 \xrightarrow{0.7} t_1 \xrightarrow{0.0} t_3 \xrightarrow{0.4} t_4 \xrightarrow{1.2} t_2 \xrightarrow{0.5} t_3 \xrightarrow{1.4} Z \]
State Space Reduction

Example

\[ \sigma = (t_1 t_3 t_4 t_2 t_3) \]

\[ m_\sigma = (1, 2, 2, 1, 1) \]
Example ( continuation )

\[ \Sigma_\sigma = \begin{pmatrix} 
  x_4 + x_5 \\ 
  x_5 \\ 
  x_5 \\ 
  x_5 \\ 
  x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ 
  \# 
\end{pmatrix} \]

and
State Space Reduction

Example ( continuation )

\[ B_\sigma = \{ \begin{align*} &0 \leq x_0, \quad x_0 \leq 2, \quad x_0 + x_1 + x_2 \leq 5 \\
&0 \leq x_1, \quad x_2 \leq 2, \quad x_2 + x_3 \leq 5 \\
&1 \leq x_2, \quad x_3 \leq 2, \quad x_0 + x_1 + x_2 + x_3 \leq 5 \\
&1 \leq x_3, \quad x_4 \leq 2, \quad x_0 + x_1 + x_2 + x_3 + x_4 \leq 5 \\
&0 \leq x_4, \quad x_5 \leq 2, \quad x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \leq 5 \\
&0 \leq x_5, \quad x_0 + x_1 \leq 5 \quad x_4 + x_5 \leq 2 \end{align*} \} . \]
Example ( continuation )

The run $\sigma(\tau)$ with

$$\sigma(\tau) := z_0 \xrightarrow{0.7} t_1 \xrightarrow{0.0} t_3 \xrightarrow{0.4} t_4 \xrightarrow{1.2} t_2 \xrightarrow{0.5} t_3 \xrightarrow{1.4} z$$

is feasible.
State Space Reduction

Example ( continuation )

The run \( \sigma(\tau) \) with

\[
\sigma(\tau) := z_0 \xrightarrow{0.7} t_1 \xrightarrow{0.0} t_3 \xrightarrow{0.4} t_4 \xrightarrow{1.2} t_2 \xrightarrow{0.5} t_3 \xrightarrow{1.4} z
\]

is feasible.
Example ( continuation )

The run $\sigma(\tau)$ with

$z_0 \xrightarrow{0.7} t_1 \xrightarrow{0.0} t_3 \xrightarrow{0.4} t_4 \xrightarrow{1.2} t_2 \xrightarrow{0.5} t_3 \xrightarrow{1.4} (m, \begin{pmatrix} 1.9 \\ 1.4 \\ 1.4 \\ 4.2 \end{pmatrix})$

is feasible.
### State Space Reduction

#### Example (continuation)

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<thead>
<tr>
<th>$\beta$</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$\Sigma_\sigma(t_1)$</th>
<th>$\Sigma_\sigma(t_2)$</th>
<th>$\Sigma_\sigma(t_5)$</th>
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</thead>
<tbody>
<tr>
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<td>0.0</td>
<td>0.4</td>
<td>1.2</td>
<td>0.5</td>
<td>1.4</td>
<td>1.9</td>
<td>1.4</td>
<td>4.2</td>
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<tr>
<td>$\beta_1$</td>
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<td>0.4</td>
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<td>0.5</td>
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### Example ( continuation )

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<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td>5.0</td>
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</table>
State Space Reduction

Example ( continuation )

Hence, the runs
\[ \sigma(\tau_1^*) := z_0 \xrightarrow{1} t_1 \xrightarrow{0} t_3 \xrightarrow{1} t_4 \xrightarrow{1} t_2 \xrightarrow{0} t_3 \xrightarrow{1} [z] \]

and
\[ \sigma(\tau_2^*) := z_0 \xrightarrow{1} t_1 \xrightarrow{0} t_3 \xrightarrow{0} t_4 \xrightarrow{2} t_2 \xrightarrow{0} t_3 \xrightarrow{2} [z] \]

are feasible in \( Z \), too.
State Space Reduction

Corollary

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Corollary

- Each feasible t-sequence $\sigma$ in $Z$ can be realized with an "integer" run.
- Each reachable marking in $Z$ can be found using "integer" runs only.
- If $z$ is reachable in $Z$, then $\lfloor z \rfloor$ and $\lceil z \rceil$ are reachable in $Z$, too.
- The length of the shortest and longest time path between two arbitrary states are natural numbers.
State Space Reduction

Definition

A state $z = (m, h)$ in a TPN is integer one iff for all enabled transitions $t$ at $m$ holds: $h(t) \in \mathbb{N}$. 

Theorem (3)

Let $Z$ be a FTPN. The set of all reachable integer states in $Z$ is finite if and only if the set of all reachable markings in $Z$ is finite.

Remark:
Theorem 3 can be generalized for all TPNs (applying a further reduction).
State Space Reduction

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Reachability Graph

Definition

**Basis)**

\[ z_0 \in RG(Z) \]
Reachability Graph

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Let \( z \) be in \( RG(Z) \) already.
1. for \( i=1 \) to \( n \) do
   if \( z \xrightarrow{t_i} z' \) possible in \( Z \) then \( z' \in RG(Z) \) end

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Quantitative Analysis of TPNs
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Reachability Graph

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\[ \implies \] The reachability graph is a weighted directed graph.
A TPN and its full Reachability Graph

Example (A TPN $Z$ and its full reachability graph $RG^{(1)}(Z)$)
Example (The reduced reachability graphs $RG^{(2)}(Z)$ and $RG(Z)$)
Example (The reachability graph \( RG(Z_3) \))
**Definition**

The transition sequence $\sigma$ is a **feasible T-invariant** in a TPN $Z$ if for each marking $m$ in $Z$ holds: $m \xrightarrow{\sigma} m$. 
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For **timeless PN**: $\sigma$ is a feasible T-invariant iff $m = m + C \cdot \psi(\sigma)$ and $\psi(\sigma)$ - the Parikh-vector of $\sigma$. $\xrightarrow{\longrightarrow}$ easy to be found.
Lemma

Let $Z$ be a TPN, $S(Z)$ be the skeleton of $Z$ and $\sigma$ be a feasible $T$-invariant in $S(Z)$.

$\sigma$ is a feasible $T$-invariant in $Z$ iff $B_\sigma$ has a solution.
Lemma

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Computing the T-invariants of a $Z$:

- Solve the linear system of equations $C \cdot x = 0$ for $x \in \mathbb{N}$. 
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Computing the T-invariants of a $Z$:

- Solve the linear system of equations $C \cdot x = 0$ for $x \in \mathbb{N}$.
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Quantitative Analysis of TPNs
**Lemma**

Let $Z$ be a TPN, $S(Z)$ be the skeleton of $Z$ and $\sigma$ be a feasible $T$-invariant in $S(Z)$.

\(\sigma\) is a feasible $T$-invariant in $Z$ iff $B_\sigma$ has a solution.

Computing the $T$-invariants of a $Z$:

- Solve the linear system of equations $C \cdot x = 0$ for $x \in \mathbb{N}$.
- Decide feasibility of a $T$-invariant $\sigma$ with $\text{Parikh}(\sigma) = x$.
- If $\sigma$ is feasible, then solve the linear system of inequalities $B_\sigma$ in $\mathbb{R}^+$.
**Remark:** The reachability graph of a TPN is not used for computing the feasible T-invariants of $Z$

$\implies$

feasible T-invariants for **unbounded** nets can be computed!
Let $Z = (P, T, F, V, I, m_0)$ be a TPN. Then the following problems can be decided/computed without knowledge of its RG:
Let $Z = (P, T, F, V, I, m_0)$ be a TPN. Then the following problems can be decided/computed without knowledge of its RG:

**Result 1:**

**Input:** The time function $I$ is fixed, $\sigma$ is an arbitrary transition sequence.

**Output:** Feasibility of $\sigma$ in $Z$?

**Solution:** Solve a linear system of inequalities in $\mathbb{R}_0^+$. 

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**Quantitative Analysis of TPNs**
Let \( Z = (P, T, F, V, I, m_0) \) be a TPN.
Then the following problems can be decided/computed without knowledge of its RG:

**Result 2:**

**Input:** The time function \( I \) is not fixed, \\
\( \sigma \) is an arbitrary transition sequence.

**Output:** Feasibility of \( \sigma \) in \( Z \) for a fixed \( I \)?

**Solution:** Solve a linear system of inequalities in \( \mathbb{Q}_0^+ \).
Let $Z = (P, T, F, V, I, m_0)$ be a TPN. Then the following problems can be decided/computed without knowledge of its RG:

**Result 3:**

**Input:** The time function $I$ is fixed, $\sigma$ is an arbitrary transition sequence.

**Output:** $\text{min} / \text{max}$-length of $\sigma$.

**Solution:** Solve a linear program in $\mathbb{R}_0^+$. (Actually, the solution is in $\mathbb{N}$.)
Let \( Z = (P, T, F, V, I, m_0) \) be a TPN. Then the following problems can be decided/computed without knowledge of its RG:

**Result 4:**

**Input:** The time function \( I \) is not fixed, \( \sigma \) is an arbitrary transition sequence, and \( \lambda \) is an arbitrary real number.

**Output:** Existence of a fixed \( I \) and a run \( \sigma(\tau) \) in \( Z \) and the length of \( \sigma(\tau) \leq \lambda \)?

**Solution:** Solve a linear program in \( \mathbb{Q}_0^+ \).
Result 5:

**Input:** The time function $I$ is not fixed, $\sigma_1 = (\sigma, t')$ is an arbitrary $t$-sequence and $\sigma_2 = (\sigma, t'')$ is an arbitrary $t$-sequence.

**Output:** Existence of a fixed $I$ so that $\sigma_1$ is feasible in $Z$ and $\sigma_2$ is not feasible in $Z$?

**Solution:** Solve

\[
\max\left\{<c', x> \mid A' \cdot x \leq b'\right\} < \min\left\{<c'', x> \mid A'' \cdot x \leq b''\right\}.
\]

linear program in $\mathbb{Q}_0^+$

linear program in $\mathbb{Q}_0^+$. 

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Quantitative Analysis of TPNs
Let \( Z = (P, T, F, V, I, m_o) \) be a bounded TPN. Additionally the following problems can be decided/computed with the knowledge of its RG, amongst others:
Let $Z = (P, T, F, V, I, m_o)$ be a bounded TPN. Additionally, the following problems can be decided/computed with the knowledge of its RG, amongst others:

**Result 6:**

**Input:** $z$ and $z'$ - two states (in $Z$).

**Output:**
- Is there a path between $z$ and $z'$ in $RG(Z)$?
- If yes, compute the path with the shortest time length.

**Solution:** By means of prevalent methods of the graph theory, e.g., Bellman-Ford algorithm (the running time is $O(|V| \cdot |E|)$) and $RG(Z) = (V, E)$.
Let $Z = (P, T, F, V, I, m_o)$ be a bounded TPN. Additionally the following problems can be decided/computed with the knowledge of its RG, amongst others:

**Result 7:**

**Input:** $m$ and $m'$ - two markings (in $Z$).

**Output:**
- Is there a path between $m$ and $m'$ in $RG(Z)$?
- If yes, compute the path with the shortest time length.

**Solution:** By means of prevalent methods of the graph theory, for computing all-pairs shortest paths. The running time is polynomial, too.
Definition

The **longest path** between two states (vertices in \( RG(Z) \)) \( z \) and \( z' \) is \( lp(z, z') \) with

\[
lp(z, z') := \begin{cases} 
\infty, & \text{if a cycle is reachable starting on } z \\
\max \sum_{\sigma(\tau)} \tau_i, & \text{if } z \xrightarrow{\sigma(\tau)} z'
\end{cases}
\]
Result 8:

**Input:** $z$ and $z'$ - two states (in $Z$).

**Output:**
- Is there a path between $z$ and $z'$ in $RG(Z)$?
- If yes, compute the path with the longest time length.

**Solution:** By means of prevalent methods of the graph theory, e.g. Bellman-Ford algorithm (polyn. running time). or by computing all strongly connected components of $RG(Z)$. (linear running time)
Result 9:

**Input:** $m$ and $m'$ - two states (in $\mathbb{Z}$).

**Output:**
- Is there a path between $z$ and $z'$ in $RG(Z)$?
- If yes, compute the path with the longest time length.

**Solution:** By means of prevalent methods of the graph theory, e.g. Bellman-Ford algorithm (polyn. running time). or by computing all strongly connected components of $RG(Z)$. (linear running time)
Transformation Timed PN $\longrightarrow$ Time PN

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Quantitative Analysis of TPNs
Conclusion

- theoretical approach

  $BN \rightarrow modelling \rightarrow PN \rightarrow \text{modelling of steady state} \rightarrow$

  $DPN \rightarrow analysing \rightarrow TPN$

- experimental approach

  $BN \rightarrow modelling \ & analysing \rightarrow TPN$