Controlling Petri Net Behavior Using Time Constraints

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CS&P 2015
Outline

1. Preliminaries: Petri nets, boundedness, liveness.
2. Problem statement and motivating examples.
3. Time Petri nets.
4. Computing intervals for making a live net also bounded.
5. Validation.
7. Conclusions.
\((N^*, m^*_0)\): a *live* Petri net with one *unbounded* place \(s_3\).

A Petri net is *live* iff every transition is potentially enabled in any reachable marking.

A run is called *feasible* iff it starts from a reachable marking.
To transform a live and unbounded Petri nets into a live and bounded one by adding some control.
Problem statement

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Important for applications in business process management (soundness = boundedness + liveness), biology, etc.
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Adding time durations to transitions (M. Heiner, 2007)

When a Petri net is covered by transition invariants, these invariants can be used for computing time durations for transitions and thus transforming a live and unbounded Petri net into a live and bounded Timed Petri net with the same structure.
Some motivating examples: time durations

[Heiner, 2006]

Live and unbounded Petri nets. The right one can be transformed into live and bounded by adding time durations.
A Petri net is *weakly bounded* iff it is unbounded, but for every reachable marking a bounded run is enabled.

[DESEL 2006], WEAKLY BOUNDED PETRI NETS; AWPN 2006
Some motivating examples

A Petri net is \textit{weakly bounded} iff it is unbounded, but for every reachable marking a bounded run is enabled.

The distinction between bounded, weakly bounded and not weakly bounded Petri nets is very important for applications.

[DESEL 2006], WEAKLY BOUNDED PETRI NETS; AWPN 2006
Let \((\mathcal{N}, m_0)\) be a live and unbounded Petri net,

We present an algorithm for finding time intervals (for transitions) that will make the net bounded, keeping its liveness.
Time Petri nets

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Controlling Petri Net Behavior Using Time
Time Petri net

\[ m_0 = (2, 0, 1) \]

\[ p\text{-marking} \]

\[ h_0 = (♯, 0, 0, 0) \]

\[ t\text{-marking} \]

\[ z = (m, h) \]

state

\[ h(t) \] is the time shown by the clock of \( t \) since the last enabling of \( t \).
\[ m_0 = (2, 0, 1) \quad p\text{-marking} \]
$m_0 = (2, 0, 1)$  \textit{p}-marking

$h_0 = (\#, 0, 0, 0)$  \textit{t}-marking
\[ m_0 = (2, 0, 1) \quad \text{p-marking} \]
\[ h_0 = (\#, 0, 0, 0) \quad \text{t-marking} \]
\[ z := (m, h) \quad \text{state} \]

\( h(t) \) is the time shown by the clock of \( t \) since the last enabling of \( t \).
Let $\mathcal{Z}$ be a TPN and let $z = (m, h)$, $z' = (m', h')$ be two states. $\mathcal{Z}$ changes from state $z = (m, h)$ into the state $z' = (m', h')$ by:

- firing a transition
- time elapsing

Notation: $z \xrightarrow{t} z'$ $z \xrightarrow{\tau} z'$
Time Petri net

\[ Z_1 : \]

\[ (m_0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}) \]
Time Petri net

\[ Z_1: \]

\[ \begin{align*}
Z_1: & \\
[1,5] & t_1 \\
\text{p} & \text{p}_1 \\
2 & \text{p}_1 \rightarrow \text{p}_2 \\
[0,3] & t_2 \\
[2,4] & t_3 \\
[2,3] & t_4 \\
\text{p}_2 & \text{p}_3 \\
\end{align*} \]

\[ (m_0, \begin{pmatrix} 0 \\ \# \\ \# \\ 0 \end{pmatrix}) \rightarrow_{1.3} (m_1, \begin{pmatrix} 1.3 \\ \# \\ \# \\ 1.3 \end{pmatrix}) \]

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Controlling Petri Net Behavior Using Time
Time Petri net

\[ z_0 \xrightarrow{1.3} (m_1, \begin{pmatrix} 1.3 \\ \\ 1.3 \end{pmatrix}) \xrightarrow{1.0} (m_2, \begin{pmatrix} 2.3 \\ \\ 2.3 \end{pmatrix}) \]
Time Petri net

\[ Z_0 \xrightarrow{1.3} 1.0 (m_2, \begin{pmatrix} 2.3 \\ 2.3 \end{pmatrix}) \xrightarrow{t_4} \]
Time Petri net

\[ Z_1: \]

\[
\begin{align*}
&\begin{array}{c}
\mathbb{P}_1 \\
\mathbb{P}_2 \\
\mathbb{P}_3
\end{array}
\end{align*}
\]

\[
\begin{align*}
t_1 & \quad \begin{array}{c}
[1,5] \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
t_2 & \quad \begin{array}{c}
[0,3] \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
t_3 & \quad \begin{array}{c}
[2,4] \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
t_4 & \quad \begin{array}{c}
[2,3] \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
2.3 \\
\# \\
2.3
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
2.3 \\
\# \\
0.0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
z_0 & \quad \begin{array}{c}
1.3 \quad 1.0
\end{array}
\end{align*}
\]

\[
\begin{align*}
(m_2, \quad \begin{array}{c}
2.3 \\
\#
\end{array})
\end{align*}
\]

\[
\begin{align*}
(t_4 \rightarrow (m_3, \quad \begin{array}{c}
2.3 \\
\#
\end{array})
\end{align*}
\]
\[ Z_1 : \]

\[ Z_0 \xrightarrow{1.3} 1.0 \xrightarrow{t_4} (m_3, \begin{pmatrix} 2.3 \\ \# \\ \# \end{pmatrix}) \xrightarrow{2.0} (m_4, \begin{pmatrix} 4.3 \\ \# \\ \# \end{pmatrix}) \]
### Notations:

- **transition sequence:** \( \sigma = t_1 \cdots t_n \),
- **run:** \( \sigma(\tau) = \tau_0 \tau_1 \cdots \tau_{n-1} t_n \tau_n \) and \( \tau_i \in \mathbb{R}_0^+ \)
- **parametric run:** \( (\sigma(\tau) = \tau_0 \tau_1 \cdots \tau_{n-1} t_n \tau_n, B_\sigma) \) and \( \tau_i \) are variables which satisfied a set of conditions \( B_\sigma \)
- **parametric state:** \( (z_\sigma, B_\sigma) \)
- **parametric state space:** \( \text{parStSp}(\mathcal{Z}) = \bigcup_{\sigma} (z_\sigma, B_\sigma) \)
- **StSp(\mathcal{Z}) = \bigcup_{\sigma} K_\sigma \) where \( K_\sigma := \{z_\sigma(\beta(x)) \mid \beta(x) \text{ is a solution of } B_\sigma\} \).
Notations:

- **transition sequence:** $\sigma = t_1 \cdots t_n$,
- **run:** $\sigma(\tau) = \tau_0 t_1 \tau_1 \cdots \tau_{n-1} t_n \tau_n$ and $\tau_i \in \mathbb{R}^+_0$
- **parametric run:** $(\sigma(x) = x_0 t_1 x_1 \cdots x_{n-1} t_n x_n, B_\sigma)$ and $x_i$ are variables which satisfied a set of conditions $B_\sigma$
- **parametric state:** $(z_\sigma, B_\sigma)$
- **parametric state space:** $\text{parStSp}(Z) = \bigcup_{\sigma} (z_\sigma, B_\sigma)$
- $\text{StSp}(Z) = \bigcup_{\sigma} K_\sigma$ where

$$K_\sigma := \{ z_\sigma(\beta(x)) \mid \beta(x) \text{ is a solution of } B_\sigma \}.$$
Time Petri net: Example

\[ K_ε = \{( (0, 1, 1), (x_0, \#, \#, x_0) ) | 0 \leq x_0 \leq 3 \}\}.

\[ K_{t_4} = \{( (1, 1, 0), (x_0 + x_1, \#, x_1, \#) ) | 2 \leq x_0 \leq 3, x_0 + x_1 \leq 5, 0 \leq x_1 \leq 4 \}\}.

The set of conditions \( B_{t_4} \) is the union of the three sets

\[ B_ε, \left\{ \text{eft}(t_4) \leq h_ε(t_4) \right\} = \{2 \leq x_0\} \] and \( \{0 \leq h_σ(t) \leq lft(t) | t^- \leq m_σ\} = \{x_0 + x_1 \leq 5, 0 \leq x_1 \leq 4\}. \]
Proposition

Let $(\mathcal{N}, m_0)$ be a live Petri net. Then there exists a feasible cyclic run, including all transitions in $\mathcal{N}$. 
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Proposition

Let $(\mathcal{N}, m_0)$ be a Petri net, and let $\mathcal{Z} = (\mathcal{N}, m_0, I)$ be a Time Petri net, obtained from $\mathcal{N}$ by adding intervals to each transition. Then the reachability tree for $\mathcal{Z}$ is a subgraph of the reachability tree for $(\mathcal{N}, m_0)$.
Stage 1. Find all minimal feasible cycles, which include all transitions.

Use the technique described in

Jörg Desel
On Cyclic Behaviour of Unbounded Petri Nets, [ACSD-2013]

Following this technique for each minimal feasible cyclic run $\sigma$ we also find a finite initial run $\tau$, such that $\tau\sigma^*$ is an initial run in $(N, m_0)$.

If $(N, m_0)$ does not have such cycles, then the problem does not have a solution.
\((N^*, m_0^*)\) has five minimal cyclic runs with all transitions. Three of them have empty prefixes, and two have prefixes \(\tau_1 = b\) and \(\tau_2 = ba\), respectively:

- \(babcda\)
- \(babcad\)
- \(babacd\)

\(\tau_1 = b\) and \(\tau_2 = ba\):

- \(abacbd\)
- \(bacbad\)
Stage 2. Construct a spine tree

A spine tree is a subgraph of a reachability tree, containing exactly all minimal feasible cyclic runs together with their prefixes.

A spine tree contains the behavior that should be saved to keep a Petri net live.
The spine tree for the net \((\mathcal{N}^*, m^*_0)\):

```
1,0,0,1,0
   b
0,1,1,1,0
   a
1,0,1,1,0
   b
0,1,2,1,0
   a
1,0,2,1,0
   c
1,0,0,0,1
   b
0,1,1,0,1
   d
1,0,1,0,1
   a
0,1,1,0,1
   d
1,0,1,0,1
   a
1,0,1,0,1
```

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Controlling Petri Net Behavior Using Time
A spine-based coverability tree is a special kind of a coverability tree that includes a spine tree as a backbone. Leafs in a spine-based coverability tree will be additionally colored with green or red. This coloring will be used then for computing time intervals.
Algorithm for computing a spine-based coverability tree:

Step 1. Start with the given spine tree. Color all leafs in green.

Step 2. Repeat until all nodes are colored:
For each uncolored node \( v \) labeled with a marking \( m \):

1. check whether there is a marking \( m' \), directly reachable from \( m \) and not included in the current tree. For each such marking \( m' \), where \( m \xrightarrow{t} m' \):
   1. Add a node \( v' \) labeled with \( m' \) as well as the corresponding arc from \( v \) to \( v' \) labeled with \( t \).
   2. If the marking \( m' \) strictly covers a marking in some node on the path from the root to \( v' \), then \( v' \) becomes a leaf and gets the red color.
   3. Otherwise, if the marking \( m' \) coincides with a marking labeling some node on the path from the root to \( v' \), then \( v' \) becomes a leaf and gets the green color.
   4. Otherwise, leave \( v' \) uncolored.

2. Color the node \( v \) in yellow.
The spine-based coverability tree for $(N_1, m_0)$:
Let $\mathcal{T}$ be a spine-based coverability tree. By the construction of $\mathcal{T}$, all its leaves are colored either in green, or red.

Add a time interval $[a_t, b_t]$ to each transition $t \in T$. All $a_t, b_t$ are unknown and have to be calculated in stage 5. Thus, we are going to construct an interval function $I : \mathcal{T} \rightarrow \mathbb{Q}_0^+ \times \mathbb{Q}_0^+$ and a TPN $(\mathcal{N}, m_0, I)$, respectively.

For this we consider every path from the root to a green leaf as a parametric run. Additionally, we forbid a branching to a red leaf using strict inequality:
(1) Let $v_g$ be a green leaf and let $\sigma$ be the path from the root to this leaf. Consider the parametric run $(\sigma(x), B_\sigma)$.
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(2) Let $v_r$ be a red leaf. We consider $\sigma$, $v_0$, $v^*$, $v_g$, $v_r$ (labeled with $m_0$, $m^*$, $m_g$, $m_r$, respectively) as follows:

$$m_0 \xrightarrow{\sigma^*} m^* \xrightarrow{\sigma_g} v_g, \quad m_0 \xrightarrow{\sigma^*} m^* \xrightarrow{t_r} \sigma_r \xrightarrow{\sigma_r} v_r, \quad \sigma = \sigma^* t_r \sigma_r,$$

where

- $\sigma$ is the path from the root to $v_r$,
- $v^*$ be the youngest ancestor of $v_r$ such that at least one run goes from $v^*$ to a green leaf $v_g$,
- add to $B_{\sigma^*}$ the constrain (strong inequality) $h_{\sigma^*}(t_r) < a_{t_r}$.  

Stage 5. Compute an interval function $I$

Solve the system of linear inequalities $B$,

$$B := \bigcup \{B_{\sigma} \mid \sigma \text{ is an initial run to a green node}\} \cup \{0 \leq a_t \leq b_t \mid t \in T\} \cup \bigcup \{h_{\sigma^*}(t_r) < a_{t_r} \mid \text{w.r.t. Stage 4}\}.$$
Example

\[
B := \begin{cases}
    a_b \leq x_0 \leq b_b \\
    a_a \leq x_1 \leq b_a \\
    a_b \leq x_2 \leq b_b \\
    a_a \leq x_3 \leq b_a \\
    a_c \leq x_3 \\
    0 \leq a_a \\
    0 \leq a_d \\
    a_b \leq x_4 \leq b_d \\
    0 \leq x_5 \\
    0 \leq x_6 \leq b_b \\
    0 \leq x_7 \leq b_d \\
    x_7 < a_b \\
    0 \leq a_b \\
    0 < a_b \\
    0 \leq x_8 \\
    x_9 < a_b \\
    0 \leq x_{10} \\
    0 \leq x_{11} \\
    a_a \leq x_{12} \leq b_a \\
    0 \leq x_{13} < a_b \\
    0 \leq x_{14} \\
    0 \leq x_{15} \\
    a_b \leq x_9 + x_{10} \\
    x_{13} + x_{14} \leq b_b \\
    a_d \leq x_{10} + x_{12} \leq b_d \\
    x_{12} + x_{15} \leq b_a
\end{cases}
\]

Subsequently, with respect to the properties of an interval function, it has to be true:

\[a_b \geq 1, \quad b_b \geq 1.\]

A solution (with minimal values) for the interval function \(I^*\) is, e.g., \(a_a = b_a = 0, a_b = b_b = 1, a_c = b_c = 0, a_d = 0\) and \(b_d = 1\).
Theorem

Let \((N, m_0)\) be a live and unbounded Petri net, for which there exists a feasible cyclic run, which includes all transitions in \(N\). Let then \(B\) be the set of inequalities constructed for \((N, m_0)\) according to the algorithm described above. If \(B\) has a solution in \(\mathbb{Q}_0^+\), then this defined an interval function \(I\) such that the Time Petri net \(Z = (N, m_0, I)\) is live and bounded.
Let consider the PN \((\mathcal{N}^*, m_0^*)\) again:

The PN \((\mathcal{N}^*, m_0^*)\) has 5 minimal cyclic runs:

- \(babcda\)
- \(babcad\)
- \(babacd\)
- \(babacbd\)
- \(babacbad\)
Let consider the PN \((\mathcal{N}^*, m_0^*)\) again:

The PN \((\mathcal{N}^*, m_0^*)\) has 5 minimal cyclic runs:

\[ babcda \quad babcad \quad babacd \quad babacbd \quad babacbad \]
Let consider the PN \((\mathcal{N}^*, m_0^*)\) again:

The PN \((\mathcal{N}^*, m_0^*)\) has 5 minimal cyclic runs:

\[babcd, babcad, babacd, babacbd, babacbad\]
Some Comparisons

Let consider the PN \((N^*, m_0^*)\) again:

1. The PN \((N^*, m_0^*)\) has 5 minimal cyclic runs:
   \[ babcda \quad babcad \quad babacd \quad babacbd \quad babacbad \]

2. The Timed PN \(D^* = (N^*, m_0^*, D^*)\) calculated using T-invariants has 1 minimal cycle:
   \[ babacbad \text{ (or more exactly: } bab \quad c \quad b \quad a \quad d ) \]
Some Comparisons

Let consider the PN \((\mathcal{N}^*, m_0^*)\) again:

1. The PN \((\mathcal{N}^*, m_0^*)\) has 5 minimal cyclic runs:
   \(babcda \ babcad \ babacd \ babacbd \ babacbad\)

2. The Timed PN \(\mathcal{D}^* = (\mathcal{N}^*, m_0^*, D^*)\) calculated using T-invariants has 1 minimal cycle:
   \(babacbad\) (or more exactly: \(bab^a b^b c^a d\))

3. The PN with priorities \(\mathcal{P}_1 = (\mathcal{N}_1, \ll, m_0)\) has 3 minimal cyclic runs: \(babcda \ babcad \ babacd\)
Some Comparisons

Let consider the PN \((N^*, m_0^*)\) again:

\[
\begin{array}{cccc}
S3 & S & S52 & S4 \\
1 & 0,1,1,0 & 1,0,2,1,0 & 1,0,0,0,1 \\
1 & 0,1,0,1 & 0,1,0,0,1 & 0,1,0,1,0 \\
1 & 0,1,1,0 & 0,1,1,1,0 & 0,1,1,0,1 \\
1 & 0,1,2,1,0 & 0,1,0,1,0 & 0,1,0,1,0 \\
1 & 0,1,0,0,1 & 0,1,0,1,0 & 0,1,0,1,0 \\
\end{array}
\]

1. The PN \((N^*, m_0^*)\) has 5 minimal cyclic runs:  
   \(babcda \ babcad \ babacd \ babacbd \ babacbad\)

2. The Timed PN \(D^* = (N^*, m_0^*, D^*)\) calculated using T-invariants has 1 minimal cycle:
   \(babacbad\) (or more exactly: \(bab \ a \ b \ a \ c \ b \ d\))

3. The PN with priorities \(P_1 = (N_1, \ll, m_0)\) has 3 minimal cyclic runs:  
   \(babcda \ babcad \ babacd\)

4. The Time PN \(Z^* = (N^*, m_0^*, l^*)\) has all 5 minimal cyclic runs.
The possibility for obtaining a live and bounded Petri net from a live and unbounded one by adding time intervals was investigated.
Conclusions

• The possibility for obtaining a live and bounded Petri net from a live and unbounded one by adding time intervals was investigated.

• Necessary conditions for existence of such intervals were obtained. This conditions are not sufficient. But ...
Conclusions

- The possibility for obtaining a live and bounded Petri net from a live and unbounded one by adding time intervals was investigated.

- Necessary conditions for existence of such intervals were obtained. This conditions are not sufficient. But ...

- An algorithm for computing time intervals for transforming a live and unbounded Petri net into live and bounded net by ascribing these intervals to transitions was developed. This algorithm guarantees that the computed interval function solve the problem, i.e. the Time Petri net is live and bounded.
Thank you!