Time Petri Nets: Theory, Tools and Applications

Part II

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Outline

1. Timed Petri Nets
   - Introduction
   - Timed Petri Nets and Turing Machines
   - State Space
   - State Equation
   - Time Petri Nets vs. Timed Petri Nets

2. Further Variations of Time Dependent Petri Nets

3. Conclusion

4. Appendix
Timed Petri Net: An Informal Introduction

Statics:

Petri Net

\[ D : \]

\[ t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_4 \]

\[ p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4 \]
Timed Petri Net: An Informal Introduction

Statics:
Timed Petri Net: An Informal Introduction

Dynamics:

\[ D_1 : \]

\[
t_1 \quad <3> \quad t_2 \quad <2> \quad <1> \quad t_3 \quad <6> \quad t_4 \quad <0>\]

\[
p_2 \quad p_1 \quad p_3 \quad p_4\]

**firing mode**: maximal step
Timed Petri Net: An Informal Introduction

Dynamics:

firing mode: maximal step
Timed Petri Net: An Informal Introduction

Dynamics:

\[ \mathcal{D}_1 : \]

firing mode: maximal step
Timed Petri Net: An Informal Introduction

Dynamics:

firing mode: maximal step
A formal definition of a Timed Petri Net can be found in the Appendix, Part II.
Remark:

The power of the Timed Petri Nets is equal to the power of the Turing Machines.
Timed Petri Nets and Counter Machines

Remark:

The power of the Timed Petri Nets is equal to the power of the Turing Machines.

Idea:

Simulation of an arbitrary Counter Machine with a Timed Petri Net.

Sufficiently: To simulate the command

\[ l:DEC(i):r:s \text{ (zero-test).} \]
Zero-test

Zero-test ($l:DEC(i):r:s$) for Timed PN with firing mode maximal step

![Timed Petri Net Diagram]
Zero-test (\(l:DEC(i):r:s\)) for Timed PN with firing mode maximal step.
Reachability graph

\[ D_1 : \]

\[
\begin{array}{c}
\text{p}_1 \\
\text{t}_1 \quad < 3 > \quad \text{t}_2 \quad < 2 > \\
\text{p}_2 \\
\text{t}_5 \quad < 1 > \quad \text{t}_6 < 6 > \quad \text{t}_3 < 0 > \quad \text{t}_4
\end{array}
\]

\[
\begin{array}{c}
\text{p}_3 \\
2
\end{array}
\]

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Reachability graph
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State Equation in classical PN

Let $\mathcal{N}$ be a classical PN with

- $m_1$ and $m_2$ two markings in $\mathcal{N}$,
- $\sigma = t_1 \ldots t_n$ a firing sequence, and
- $m_1 \xrightarrow{\sigma} m_2$.

Then it holds:

$$m_2 = m_1 + C \cdot \pi_\sigma,$$

(state equation)

where $C$ is the incidence matrix of $\mathcal{N}$ and $\pi_\sigma$ is the Parikh vector of $\sigma$. 
State Equation in classical PN

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In each PN $\mathcal{N}$ with initial marking $m_0$ it holds: If $m \neq m_0 + C \cdot \pi_\sigma$ then $m$ is not reachable in $\mathcal{N}$. 
Extended Form of a Place Marking

\[ \mathbf{m} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{p} \]

Extended form of the \( P \)-markings after 0 1 2 3 4 5 6 time units.
Extended Form of a Place Marking

\[
m = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

extended form of the \( p\)-markings \( m \)
Extended Form of a Place Marking

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extended form of the \( p \)-markings \( m \)

after 0 1 2 3 4 5 6 time units

\( \mathcal{D}_1 \):

\( p_1 \)
\( p_2 \)
\( p_3 \)
\( p_4 \)
Theorem

Let $\mathcal{D}$ be a Timed Petri Net, $z^{(0)}$ be the initial state in extended form and

\[ z^{(0)} \xrightarrow{\mathcal{G}_1} \hat{z}^{(1)} \xrightarrow{1} \check{z}^{(1)} \xrightarrow{\mathcal{G}_2} \hat{z}^{(2)} \xrightarrow{1} \ldots \xrightarrow{\mathcal{G}_n} z^{(n)} \]

be a firing sequence ($\mathcal{G}_i$ is a multiset for each $i$). Then, it holds:

\[ m^{(n)} = m^{(0)} \cdot R^{n-1} + C \cdot \Psi_\sigma. \]

State equation
Timed Petri Nets

State Equation

\[ z^{(0)} \xrightarrow{\mathcal{G}_1} \hat{z}^{(1)} \xrightarrow{1} \tilde{z}^{(1)} \xrightarrow{\mathcal{G}_2} \hat{z}^{(2)} \xrightarrow{1} \ldots \xrightarrow{\mathcal{G}_n} z^{(n)} \]

\[ m^{(n)} = m^{(0)} \cdot R^{n-1} + C \cdot \Psi_\sigma. \quad \text{State equation} \]

- \( m^{(n)} \) and \( m^{(0)} \) are place markings in extended form.
- \( R \) is the progress matrix for \( \mathcal{D} \).
- \( C \) is the incidence matrix of \( \mathcal{D} \) in extended form.
- \( \Psi_\sigma \) is the Parikh matrix of the sequence \( \sigma = \mathcal{G}_1 \mathcal{G}_2 \ldots \mathcal{G}_n \) of multisets of transitions.
Transformation Timed PN $\rightarrow$ Time PN
If in the Timed PN a firing duration is zero, then some problems are possible:
Transformation Timed PN $\rightarrow$ Time PN

If in the Timed PN a firing duration is zero, then some problems are possible:

It is possible that both
- the set of the reachable $p$-markings and
- the set of firing sequences
in the derived TPN are supersets of the corresponding sets in the Timed PN.
Sufficient Conditions for the Nonreachability of $p$-markings

The $p$-marking $m$

- does not satisfy a state equation.
- does not satisfy the maximality condition for the firing rule.
Duration Interval Petri Nets

\[ \mathcal{DI}: \]

\begin{align*}
  &t_1 &<1,1> &p_2 \\
  &<1,3> &t_2 &<5,6> &t_3 &<1,2> &t_4 \\
  &p_1 &\Rightarrow &p_2 &\Rightarrow &p_3 &\Rightarrow &p_4 \\
\end{align*}

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Transformation Timed PN → Time PN

DI PN

< d,c >

Time PN

[0,0] [d,c]

P₁ Pₙ Q₁ Qₘ

P₁ Pₙ Q₁ Qₘ

Aₜ Bₜ

[0,0] [d,c]
Further Variations of Time Dependent Petri Nets

Petri Nets with Time Dependent Places

This class of time dependent Petri Nets is equivalent to the classical Petri Nets (and therefore not equivalent to Turing Machines).

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This class of time dependent Petri Nets is equivalent to the classical Petri Nets (and therefore not equivalent to Turing Machines).
**Theorem:** Let \( \mathcal{P} \) be a PN with time dependent places and \( T \) be the set of its transitions. Let

\[
\sigma(\tau) = \tau_0 t_1 \tau_1 t_2 \tau_2 \cdots \tau_{n-1} t_n
\]

be a feasible run in \( \mathcal{P} \) with \( \tau_i \in \mathbb{R}_0^+ \), for all \( i, 0 \leq i \leq n - 1 \). Than there exists a feasible run

\[
\sigma(\tau^*) = \tau_0^* t_1^* \tau_1^* t_2^* \tau_2^* \cdots \tau_{n-1}^* t_n
\]

in \( \mathcal{P} \) and \( \tau_i^* \in \mathbb{N} \), for all \( i, 0 \leq i \leq n - 1 \).
Further Variations of Time Dependent Petri Nets

Reachability Graph (Segment)
Given: Time dependent Petri Net

Aim: Analysis of the time dependent Petri Net

Problem: Infinite (dense) state space, TM-Equivalence

Solution:
- Parametrisation and discretisation of the state space.
- Definition of an reachability graph.
- Structurally restricted classes of time dependent Petri Nets.
- Time dependent state equation.
Thank you!
Thank you!
Thank you!
Appendix

Timed Petri Net – Statics

Definition (Timed Petri Net)

The 6-tupel $\mathcal{D} = (P, T, F, V, m_0, D)$ is called Timed Petri Net (short: DPN), iff

- the 5-tuple $(P, T, F, V, m_0) =: S(\mathcal{D})$ is a Petri Net
- $D : T \rightarrow \mathbb{Q}_0^+$, called duration function.
A pair $z = (m, u)$ is called a state in the DPN $D$ iff

- $m$ is a marking in $S(D)$ and
- $u : T \rightarrow \mathbb{R}_0^+$ with

$$\forall t \left( (t \in T \land t^- \leq m) \rightarrow u(t) \leq D(t) \right).$$
Timed Petri Net – Dynamics

Definition (maximal step)

Let \( z = (m, u) \) be a state in the DPN \( D \) and let \( T \) be the set of its transitions. Then, the set \( M \) is called a maximal set in \( z \) iff

1. \( M \subseteq T \),
2. \( \forall t \,( t \in M \implies u(t) = 0 ) \),
3. \( \sum_{t \in M} t^- \leq m \),
4. \( \forall \hat{t} \,( \hat{t} \in T \wedge \hat{t} \not\in M \wedge \hat{t}^- \leq m \wedge u(\hat{t}) = 0 ) \implies ( \sum_{t \in M} t^- + \hat{t}^- \not\leq m ) \).
Timed Petri Net – Dynamics

Definition (firing)

Let \( z_1 = (m_1, u_1) \) be a state in the DPN \( \mathcal{D} \) and let \( M \subseteq T \) holds. Than \( M \) can fire in \( z_1 \) (denoted by: \( z_1 \xrightarrow{M} \)) iff \( M \) a maximal step in \( z_1 \).

After firing of \( M \) the DPN \( \mathcal{D} \) is in the state \( z_2 = (m_2, u_2) \) (denoted by: \( z_1 \xrightarrow{M} z_2 \)) with:

\[
(1) \quad m_2 := m_1 - \sum_{t \in M} t^- + \sum_{t \in M, D(t) = 0} t^+,
\]

\[
(2) \quad u_2(t) := \begin{cases} 
D(t), & \text{if } t \in M \\
 u_1(t), & \text{else}
\end{cases}
\]

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**Definition (time elapsing)**

Let $z_1 = (m_1, u_1)$ be a state in the DPN $\mathcal{D}$. Then one time unit can elapse in $\mathcal{D}$ (denoted by $z_1 \xrightarrow{1} z_2$) iff

$$\forall t \left( (t \in T \land u_1(t) = 0) \implies t^- \not\leq m_1 \right).$$

After the elapsing of one time unit the DPN $\mathcal{D}$ is in the state $z_2 = (m_2, u_2)$ (denoted by $z_1 \xrightarrow{1} z_2$) with:

1. $m_2 := m_1 + \sum_{t \in T, \; u_1(t) = 1} t^+$,

2. $u_2(t) := \begin{cases} u_1(t) - 1, & \text{if } u_1(t) \geq 1 \\ 0, & \text{else} \end{cases}$.
Definition (incidence matrix)

Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri Net. The matrix

$$\mathbf{C}_{\mathcal{N}} := \left( - t_j^- (p_i) + t_j^+ (p_i) \right), \quad i = 1 \ldots |P|, \ j = 1 \ldots |T|$$

is called the **incidence matrix** of $\mathcal{N}$. 
Definition (Parikh vektor)

Let $\mathcal{N} = (P, T, F, V, m_0)$ be a PN and $\sigma = t_1 \ldots t_n$ be a firing sequence in $\mathcal{N}$. The vector $\pi \in \mathbb{N}^{|T|}$ with

$$\pi(t) := \text{number of appearances of the transition } t \text{ in the sequence } \sigma$$

is called the Parikh vector of $\sigma$. 
The Progress Matrix for $\mathcal{D}_1$

\[
R_{\mathcal{D}_1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

$R_{\mathcal{D}_1}$ is a $d \times d$ matrix and $d = 7 = \max \text{ duration in } R_{\mathcal{D}_1} + 1$

max. duration in $R_{\mathcal{D}_1} + 1 = 6$
The Incidence Matrix of $D_1$ in Extended Form

\[ C_{D_1} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]
The Bag-Matrix of a (global) step in a Timed PN

The matrix $G^{(i)}$ is the **bag-matrix** of the (global) firing step $\mathcal{G}_i$ iff

$$G^{(i)} = \begin{pmatrix} G_{(1)} \\ G_{(2)} \\ \vdots \\ G_{(|T|)} \end{pmatrix}, \quad G_{(s)} = \kappa_{s}^{(i)} \cdot E_{d}, \text{ where}$$

$\kappa_s^{(i)}$ is the number of appearance of $t_s$ in $\mathcal{G}_i$ and $E_d$ is the unit matrix of the dimension $d$. 
The Bag-Matrix of a (global) step in a Timed PN

The Bag-Matrix of the (global) step $G_1 = \{t_2, t_3\}$ in $D_1$

| $D_1$ is $d = 7$ and $|T| = 5$. |
|----------------------------------|
| $G_1$ is $\kappa_1^{(1)} = \kappa_4^{(1)} = \kappa_5^{(1)} = 0$ and $\kappa_2^{(1)} = \kappa_3^{(1)} = 1$. |

Finally,

$$G^{(1)} = \begin{pmatrix} 0 \cdot E_7 \\ 1 \cdot E_7 \\ 1 \cdot E_7 \\ 0 \cdot E_7 \\ 0 \cdot E_7 \end{pmatrix}$$
The Parikh Matrix of a $\sigma$ in a Timed PN

$\Psi$ is the **Parikh Matrix** of the sequence $\sigma = \mathcal{G}_1 \mathcal{G}_2 \ldots \mathcal{G}_n$ of (global) steps, i.e.

\[
\begin{align*}
\mathbf{z}^{(0)} \xrightarrow{\mathcal{G}_1} \mathbf{z}^{(1)} \xrightarrow{1} \mathbf{z}^{(1)} \xrightarrow{\mathcal{G}_2} \mathbf{z}^{(2)} \xrightarrow{1} \ldots \xrightarrow{\mathcal{G}_n} \mathbf{z}^{(n)}
\end{align*}
\]

iff:

\[
\Psi_\sigma := \sum_{i=1}^{n} G^{(i)} \cdot R^{n-i}
\]