2.4 Die Stirling Formel

Satz 3 *Es gilt*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Beweis: Die Aussage des Satzes ist äquivalent zu

$$\ln n! \sim \ln\sqrt{2\pi} + (n+\frac{1}{2})\ln n - n.$$

Sei

$$d_n := \ln n! - (n+\frac{1}{2})\ln n + n.$$

Es genügt zu zeigen,

$$\lim_{n\to\infty} d_n = \ln\sqrt{2\pi}.$$

Wir schätzen jetzt die Differenz $d_n - d_{n+1}$ ab, daraus dann das Verhalten der Folge $\{d_n\}$ und versuchen dann den Grenzwert zu bestimmen.

Es gilt

$$
\begin{aligned}
d_n - d_{n+1} &= \ln n! - \ln(n+1)! \\
&\quad -(n+\frac{1}{2})\ln n + (n+1+\frac{1}{2})\ln(n+1) \\
&\quad +n-(n+1) \\
&= \ln\frac{n!}{(n+1)!} + (n+\frac{1}{2})(\ln(n+1)-\ln n) \\
&\quad +\ln(n+1)-1 \\
&= -\ln(n+1) + (n+\frac{1}{2})\ln\frac{n+1}{n} + \ln(n+1) - 1 \\
&= \frac{2n+1}{2}\ln\frac{n+1}{n} - 1 \\
&= (2n+1)\cdot\frac{1}{2}\ln\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} - 1.
\end{aligned}
$$

Es gilt für $-1 < x < 1$:

$$\begin{aligned}
\ln(1+x) &= \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i} \\
\ln(1-x) &= \sum_{i=1}^{\infty}(-1)^{i+1}\frac{(-x)^i}{i} \\
\ln\frac{1+x}{1-x} &= \ln(1+x)-\ln(1-x) = 2\sum_{i=0}^{\infty}\frac{x^{2i+1}}{2i+1}
\end{aligned}$$

Setzen $x := \frac{1}{2n+1}$ und erhalten ($x \neq 0$)

$$\begin{aligned}
d_n - d_{n+1} &= \frac{1}{x} \cdot \frac{1}{2} \cdot 2\Big(x + \sum_{i=1}^{\infty} \frac{1}{2i+1} x^{2i+1}\Big) - 1 \\
&= \sum_{i=1}^{\infty} \frac{1}{2i+1} \frac{1}{(2n+1)^{2i}} \\
&< \sum_{i=1}^{\infty} \frac{1}{3} \frac{1}{(2n+1)^{2i}} \\
&= \frac{1}{3}\Big(\frac{1}{1-q} - 1\Big) \quad \text{wobei} \quad q = \frac{1}{(2n+1)^2} \\
&= \frac{1}{3((2n+1)^2 - 1)}
\end{aligned}$$

Offenbar gilt auch

$$\frac{1}{3(2n+1)^2} = \sum_{i=1}^{1} \frac{1}{2i+1} \cdot \frac{1}{(2n+1)^{2i}} < d_n - d_{n+1},$$

also

$$\frac{1}{3(2n+1)^2} < d_n - d_{n+1} < \frac{1}{3((2n+1)^2 - 1)}.$$

Wegen

$$
\begin{aligned}
\frac{1}{3((2n+1)^2-1)} &= \frac{1}{12n(n+1)} = \frac{1}{12n} - \frac{1}{12(n+1)} \\
\frac{1}{3(2n+1)^2} &= \frac{1}{12n(n+1)+3} = \frac{12}{12(12n(n+1)+3)} \\
&= \frac{12}{12\cdot 12n(n+1)+36} \\
&> \frac{12}{12\cdot 12n^2+12\cdot 12n+24n+13} \\
&= \frac{12}{12\cdot 12n^2+12\cdot 14n+13} \\
&= \frac{12}{(12n+1)(12n+13)} \\
&= \frac{1}{12n+1} - \frac{1}{12(n+1)+1}
\end{aligned}
$$

Beide Ungleichungen zusammen

$$\frac{1}{12n+1} - \frac{1}{12(n+1)+1} < d_n - d_{n+1} < \frac{1}{12n} - \frac{1}{12(n+1)}$$

$$(d_n - \frac{1}{12n}) - (d_{n+1} - \frac{1}{12(n+1)}) < 0 <$$
$$(d_n - \frac{1}{12n+1}) - (d_{n+1} - \frac{1}{12(n+1)+1})$$

$\Rightarrow$ Folge $\{d_n - \frac{1}{12n+1}\}$ monoton fallend

Folge $\{d_n - \frac{1}{12n}\}$ monoton wachsend.

Beide Folgen haben denselben Grenzwert $c := \lim d_n$,

$$\begin{aligned} d_n - \frac{1}{12n} \quad &< c < \quad d_n - \frac{1}{12n+1} \\ c + \frac{1}{12n+1} \quad &< d_n < \quad c + \frac{1}{12n} \end{aligned}$$

Erinnerung:

$$\begin{aligned} d_n &= \ln n! - (n + \frac{1}{2}) \ln n + n \\ \Rightarrow e^{d_n} &= n! \left(\frac{n}{e}\right)^{-n} n^{-\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
e^{c+\frac{1}{12n+1}} &< e^{d_n} < e^{c+\frac{1}{12n}} \\
e^c e^{\frac{1}{12n+1}} &< n!\left(\frac{n}{e}\right)^{-n} n^{-\frac{1}{2}} < e^c e^{\frac{1}{12n}} \\
e^c \sqrt{n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} &< n! < e^c \sqrt{n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}
\end{aligned}
$$

Bleibt zu zeigen

$$e^c = \sqrt{2\pi}.$$

Hilfsrechnungen

$$I_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$
\begin{aligned}
I_n &= \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x \, dx \\
&= \sin^{n-1} x \cdot (-\cos x)\Big|_0^{\pi/2} - \\
&\quad \int_0^{\pi/2} (n-1) \sin^{n-2} \cos x \cdot (-\cos x) \, dx \\
&= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) \, dx \\
&= (n-1)(I_{n-2} - I_n) \\
I_n &= \frac{n-1}{n} I_{n-2} \\
I_0 &= \frac{\pi}{2} \qquad\qquad\qquad I_1 = 1 \\
I_2 &= \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2} \qquad I_3 = \frac{2}{3} I_1 = \frac{2}{3}
\end{aligned}
$$

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}$$

$$I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$0 < x < \tfrac{\pi}{2}$$

$$\Rightarrow \quad 0 < \sin x < 1$$

$$\Rightarrow \quad \sin^{2n-1} x > \sin^{2n} x > \sin^{2n+1} x$$

$$\Rightarrow \quad I_{2n-1} > I_{2n} > I_{2n+1}$$

$$\Rightarrow \quad \frac{I_{2n-1}}{I_{2n+1}} > \frac{I_{2n}}{I_{2n+1}} > 1$$

$$\Rightarrow \quad \frac{2n+1}{2n} > \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdots(2n-1)\cdot(2n+1)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots(2n)\cdot(2n)} \cdot \frac{\pi}{2} > 1$$

$$\Rightarrow \quad \lim \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdots(2n-1)\cdot(2n+1)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots(2n)\cdot(2n)} \cdot \frac{\pi}{2} = 1$$

$$\Rightarrow \quad \frac{\pi}{2} = \lim \Big(\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\Big)^2 \cdot \frac{1}{2n+1}$$

$$= \lim \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)}$$

$$
\begin{aligned}
n! &= e^c\sqrt{n}n^n e^{-n} e^{\alpha_n} \\
(2n)! &= e^c\sqrt{2n}2^{2n} n^{2n} e^{-2n} e^{\beta_n}
\end{aligned}
$$

wobei $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$.

Einsetzen oben liefert

$$e^c = \sqrt{2\pi}.$$