Petri Nets with Time Windows: A Comparison with Classical Petri Nets

Jan-Thierry Wegener
Department of Mathematics, Humbold University, Berlin, Germany
wegener@math.hu-berlin.de

Louchka Popova-Zeugmann
Department of Computer Science, Humboldt University, Berlin, Germany
popova@informatik.hu-berlin.de

Extended Abstract

In this paper we present a Petri net with time restrictions at the places and compare these time dependent Petri nets with classical (timeless) Petri nets.

Petri nets with time windows (tP-PN) are derived from classical Petri nets. Additionally, each place p is associated with a time interval $[l_p, u_p]$. When a token arrives in a place p, it can not leave p before l_p time units have elapsed. During the time interval (window) $[l_p, u_p]$ the token can leave p. At the end of the interval there is not a force for leaving. When the token remains longer in the place p as u_p time units then the current time of the token in the place p is reset modulo u_p. When t becomes enabled, it can fire when enough tokens in its input places can leave them. In other words: t can fire if t is enabled and all time windows of enough tokens in all input places of t are "open". The firing itself of a transition takes no time. The time is designed by real numbers, but the interval bounds are nonnegative rational numbers. It is easy to see that w.l.o.g. the interval bounds can be considered as integers only. Thus, the interval bounds l_p and u_p of any place p are natural numbers, including zero and $l_p \leq u_p$ or $u_p = \infty$.

Every possible situation in a given tP-PN can be described completely by a time marking M with $M(p) \in (\mathbb{R}_0^+)^*$ for each place p. In general, each tP-PN has infinite number of time markings. Thus the central problem for analysis of a certain tP-PN is the knowing of its state space.

1 Introduction

1.1 Basic Notations

As usual, we use the following notations in this paper: $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^+ := \mathbb{N}\backslash\{0\}$. $\mathbb{Q}_0^+$, respectively $\mathbb{R}_0^+$, is the set of nonnegative rational numbers, and respectively the set of nonnegative real numbers. T^* denotes the language of all words over the alphabet T, including the empty word ε ; $l(\omega)$ is the length of the word ω.

1.2 Statics

Definition 1. *The pair $\mathcal{P} = (\mathcal{N}, \mathcal{I})$ is called* ***Petri net with time windows*** *(short: tP-PN) if*

- $\mathcal{N} = (P, T, F, V, m_0)$ *is a classical Petri net (short: PN)*
- $\mathcal{I} : P \to \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\})$ *with* $\mathcal{I}(p) = (l_p, u_p)$ *and* $l_p \leq u_p$ *for all places* $p \in P$.

The Petri net $S(\mathcal{P}) := \mathcal{N}$ is called the skeleton of $\mathcal{P}$.

Example 1. *In figure 1 we can see an example of a tP-PN $\mathcal{P}_1$.*

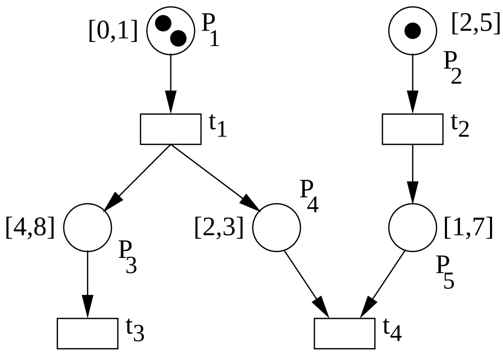

Figure 1: Example for a tP-PN.

Definition 2. *Let $\mathcal{P}$ be a tP-PN. The function $M : P \to (\mathbb{R}_0^+)^*$ is called a **time marking** of $\mathcal{P}$.*

Definition 3. *Let $\mathcal{P}$ be a tP-PN and let m_0 be the initial marking of $S(\mathcal{P})$. Then M_0 is the **initial marking** of $\mathcal{P}$ with*

$$M_0 := \begin{cases} \varepsilon & , \text{ iff } m_0(p) = 0 \\ 0^{m_0(p)} & , \text{ otherwise} \end{cases} .$$

Remark 1. *A token can only have the time 0 after initialization and, as shown below, after a transition has fired.*

Example 2. *The initial marking M_0 for $\mathcal{P}_1$ (see Example 1) is $M_0 = (0, 0; 0; \varepsilon; \varepsilon; \varepsilon)$.*

Let $\mathcal{P}$ be a tP-PN and M a time marking of $\mathcal{P}$. Furthermore let P be the set of places in $\mathcal{P}$. Then obviously, the marking $m_M := (l(M(p_1), \ldots, l(M(p_{|P|}))))$ is a marking in $S(\mathcal{P})$.

1.3 Dynamics

Definition 4. *Let M be a time marking and let t be a transition in the tP net $\mathcal{P}$. Furthermore let $M(p) = a_1^p \ldots a_n^p$ be the time marking for the place p with $a_1^p \geq a_2^p \geq \ldots \geq a_n^p$. Then t is **ready to fire** in M if*

- *t is enabled at M, i.e. $t^- \leq m_M$*
- *$\forall p (p \in {}^\bullet t \longrightarrow \forall j (j \in \{1, \ldots, t^-(p)\} \longrightarrow l_p \leq a_j^p \leq u_p))$.*

Definition 5. *Let $\mathcal{P}$ be a tP-PN and let T be the set of transitions and M a time marking in $\mathcal{P}$. A transition t can **fire** at the marking M if t is ready to fire at M. After firing t the tP-PN $\mathcal{P}$ is in the marking M':*

Let $M(p) = a_1^p \ldots a_n^p$ and $t^-(p) = k$ and $t^+(p) = r$. Then we have

$$M'(p) := \begin{cases} a_{k+1}^p \ldots a_n^p 0^r & , \text{ iff } k < n \\ 0^r & , \text{ iff } k = n \end{cases} .$$

We denote this with $M \xrightarrow{t} M'$.

Definition 6. *Let $\mathcal{P}$ be a tP-PN and M its time marking. Let $\tau \geq 0$ be a real number. Then time τ can elapse in $\mathcal{P}$ in the marking M. The net is then in the time marking M':*

Let $M(p) = a_1^p \dots a_n^p$. Let be $j \in \mathbb{N}$ with $1 \leq j \leq n$ with $u_p < a_j + \tau$ but $a_{j+1} + \tau \leq u_p$. Then we have $M'(p) = b_1^p \dots b_n^p$ with

$$b_j^p := \begin{cases} a_{j+k}^p + \tau & , \textit{iff } j + k \leq n \\ (a_{j+k}^p + \tau) \ \widehat{\bmod} \ u_p & , \textit{otherwise} \end{cases}.$$

Where

$$a \ \widehat{\bmod} \ b := \begin{cases} a \mod b & , \textit{iff } a \mod b \neq 0 \\ b & , \textit{iff } a \mod b = 0 \end{cases}.$$

A token can only have the time zero after its “new arriving” in a place. It is easy to see that the time zero of a token is equivalent to the time u_p by the definition of $\widehat{\bmod}$ in an arbitrary tP-PN.

2 Properties

In this section we compare tP-PNs and classical Petri nets with respect to reachability and liveness.

The reachability behaviour of a tP-PN is the same as of its skeleton. It is easy to see, that the power of the tP-PNs is the same as the power of the classical PN and therefore they are not equivalent to the Turing machines.

The liveness behaviour of an arbitrary tP-PN differs from the liveness behaviour of its skeleton.

2.1 Reachability

Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net. A **transition sequence** is called an arbitrary word $\sigma \in T^*$.

A **firing sequence** $\sigma = t_1 t_2 \dots t_n$ in $\mathcal{N}$ is a feasible transition sequence, i.e. there is a marking m' so that $m_0 \xrightarrow{\sigma} m'$ in $\mathcal{N}$.

A sequence $\sigma = t_1 t_2 \dots t_n$ is called **firing sequence** in a tP-PN $\mathcal{P}$ if there exists a time marking M' such that $M_0 \xrightarrow{\tau_1 t_1 \tau_2 t_2 \dots \tau_n t_n} M'$. In this case the sequence $\sigma(\tau) = \tau_1 t_1 \tau_2 t_2 \dots \tau_n t_n$ is called a **feasible run** in $S(\mathcal{P})$.

Theorem 1. *Let $\mathcal{P}$ be a tP net and $S(\mathcal{P})$ its skeleton. Then the firing sequence σ is a firing sequence in $S(\mathcal{P})$ if and only if σ is a firing sequence in $\mathcal{P}$.*

2.2 Liveness

Remark 2. *When a tP-PN $\mathcal{P}$ net is live then $S(\mathcal{P})$ is live as well. The opposite does not hold in general.*

Proof. This small example shows the tP-PN $\mathcal{P}_2$. It is obvious that $S(\mathcal{P}_2)$ is live. If we assume the sequence t_1 5.0 t_1 then it is pretty easy to see that t_2 can not become ready to fire.

$\square$

Remark 3. *Let $\mathcal{P}$ be a tP-PN with $V(f) = 1$ for each $f \in F$ and $1 \leq |^\bullet t|$. Then it is true: $\mathcal{P}$ is live iff $S(\mathcal{P})$ is live.*

Remark 4. *Note that restricting the net by $1 \leq |^\bullet t|$ is essential as figure 3 shows.*

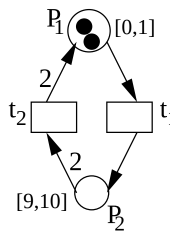

Figure 2: The timeless net is live but in the tP net the transition t_2 may not be able to fire.

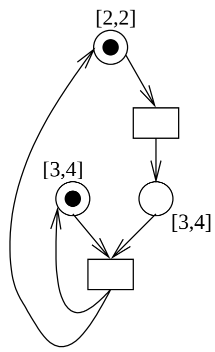

Figure 3: The timeless net is live but the tP-PN may not.

Lemma 1. *Let $\mathcal{P}$ be a tP-PN and M a time marking in $\mathcal{P}$, t an arbitrary transition in $\mathcal{P}$ and t enabled in M. Furthermore, let $1 \leq |^\bullet t|$ and let the following estimate hold for each p with $(p,t) \in F$:*

$$l_p \leq u_p \left(1 - \frac{\max\limits_{t \in T} V(p,t) - 1}{|M(p)|} \right).$$

Then t can become ready to fire in M.

Theorem 2. *Let $\mathcal{P}$ be a tP-PN and let $S(\mathcal{P})$ be its skeleton so that $1 \leq |^\bullet t|$ for every $t \in T$ and so that $S(\mathcal{P})$ is live. Furthermore let the following estimate hold for all places $p \in P$:*

$$l_p \leq \frac{u_p}{\max\limits_{t \in T} V(p,t)}.$$

Then $\mathcal{P}$ is live.

Remark 5. *Theorem 2 gives us a sufficient condition only. It is easy to find an example for a net that violates the conditions in the theorem but still is live.*

Corollary 1. *Let $\mathcal{P}$ be a tP-PN and let $S(\mathcal{P})$ be live. Let $1 \leq |^\bullet t|$ for every $t \in T$. Furthermore let for each place $p \in P$ one of the following conditions be true*

i) $l_p = 0$

ii) $u_p = \infty$

Then $\mathcal{P}$ is live too.

3 Conclusion

In this paper we have presented a PN with time restrictions at the places. Usually time dependent PN are equivalent to the Turing machins. However, we have shown that the power of this class of time dependent PNs is equivalent to the power of the classical PNs (and therefore, it is not equivalent to the Turing machine) and the reachability is the same it has a different liveness behaviour.

For a restricted class of nets we could show that the liveness behaviour can be the same. The examples in this paper also show that without the restriction given in theorem 2 further research has to be done.

Furthermere we surmise, that the following property is true: Let $\mathcal{P}$ be an arbitrary tp-PN and let $S(\mathcal{P})$ be live and let the following estimate be true for all places $p \in P$:

$$l_p \leq \frac{u_p}{\sum\limits_{t \in p^\bullet} V(p,t)}.$$

Then $\mathcal{P}$ is live.