Petri Nets with Time Windows: Possibilities and Limitations

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Abstract. In this paper we present a time extension of Petri nets called Petri net with Time Windows (short: tw-PN) where time intervals (windows) are associated with the places. We give a formal definition for this class and compare these time dependent Petri nets with their (timeless) skeletons. In particular, we compare their sets of reachable markings and their liveness behaviour. The sets of reachable markings are equal but the liveness behaviors are different. For a restricted class of tw-PNs, we give a sufficient condition for liveness equivalence. We prove that tw-PNs are not Turing equivalent and finally show the existence of runs with time gaps in a tw-PN.

1 Introduction

The Petri Nets are well-known for modelling and analysing concurrent systems where the time is implicitly involved as a causal correlation. However the Petri Nets can be extended to include time in different ways. Petri nets with time windows (tw-PNs) are derived from Petri nets (PNs) where each place p is associated with a time interval $[l_p, u_p]$. When a token arrives in a place p, it cannot leave p before l_p time units have elapsed. During the time interval (window) $[l_p, u_p]$ the token can leave p when no older (modulo u_p) tokens are in p. There is not a force for leaving at the end of the interval. When the token remains longer in the place p as u_p time units then the current time of the token in the place p is reset modulo u_p. When t becomes enabled, it can fire when enough tokens in its input places can leave them. In other words, t can fire if t is enabled and all time windows of enough tokens in all input places of t are "open". The firing itself of a transition takes no time. The time is designed by non-negative real numbers, but the interval bounds are non-negative rational numbers. It is easy to see that w.l.o.g. the interval bounds can be considered as integers only. Thus, the interval bounds l_p and u_p of any place p are natural numbers, including zero and $l_p \leq u_p$ or $u_p = \infty$.

Every possible situation in a given tw-PN can be described completely by a time marking M with $M(p) \in (\mathbb{R}_0^+)^*$ for each place p. Thus, a time marking is a vector of words over $\mathbb{R}_0^+$. In general, each tw-PN has infinite number of time markings.

The tw-PN was first introduced in [7] and later applied for modelling and diagnosis in the automation engineering in [12].

Related Work. In classical Petri nets the time is only implicitly involved in a causal context. Merlin's [15] definition of Time Petri nets (TPN) and Ramchandany's [16] definition of Timed Petri nets began a new branch of Petri nets – the time dependent Petri nets. Since this time, a great amount of different kinds of time associations have been defined. Time can be added to transitions ([15], [16], [14]), to places ([17], [7], [12]) and to edges ([3], [1]) in varied ways. The main difference between these time dependent PNs and the tw-PNs is the firing mode. An enabled transition in a tw-PN is never forced to fire, neither immediately after it has been enabled nor at any time later. A collection of well known classes of time dependent Petri nets is given in [18]. However, most of them are equivalent to the Turing machines and thus the most interesting problems like the reachability problem and the liveness problem are undecidable.

Outline of the paper. This paper is organized as follows. The second section introduces the formal definition for the tw-PN, recalls some basic definitions and remarks. In the third section we show that the set of reachable markings of a tw-PN is equal to the set of reachable markings in their skeletons. Afterwards, we prove that tw-PNs are not equivalent to Turing machines. The fourth section deals with feasible runs in a tw-PN. We show the existence of runs with “time gaps”. Furthermore, we show that a tw-PN where the time elapsings are natural numbers has a different behavior as the same net using real numbers for the time elapsings. In the fifth section we show that the liveness behavior of a tw-PN is, in general, different to the liveness behavior of its skeleton. Finally we summarize our work and give an outlook for our future work.

2 Basic Notations and Definitions

As usual, we use the following notations in this paper: $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^+ := \mathbb{N}\backslash\{0\}$. $\mathbb{Q}_0^+$ is the set of non-negative rational numbers and $\mathbb{R}_0^+$ the set of non-negative real numbers. T^* denotes the language of all words over the alphabet T, including the empty word ε ; $l(\omega)$ is the length of the word ω.

Definition 1 (classical Petri net). *The structure $\mathcal{N} = (P, T, F, V, m_o)$ is called a (classical) Petri net (short: PN) iff*

(i) P, T, F are finite sets with
$P \cap T = \emptyset$, $P \cup T \neq \emptyset$, $F \subseteq (P \times T) \cup (T \times P)$ and $dom(F) \cup cod(F) = P \cup T$
(ii) $V : F \longrightarrow \mathbb{N}^+$ (weight of the arcs)
(iii) $m_o : P \longrightarrow \mathbb{N}$ (initial marking)

A marking of a PN is a (total) function $m : P \longrightarrow \mathbb{N}$, such that $m(p)$ denotes the number of tokens at the place p. The pre-sets and post-sets of a transition t are given by $^\bullet t := \{p \mid p \in P \wedge (p,t) \in F\}$ and $t^\bullet := \{p \mid p \in P \wedge (t,p) \in F\}$, respectively. Each transition $t \in T$ induces the markings t^- and t^+, defined as follows:

$$t^-(p) = \begin{cases} V(p,t) & \text{,iff} \quad (p,t) \in F \\ 0 & \text{,iff} \quad (p,t) \notin F \end{cases} \qquad t^+(p) = \begin{cases} V(t,p) & \text{,iff} \quad (t,p) \in F \\ 0 & \text{,iff} \quad (t,p) \notin F \end{cases}$$

Moreover, Δt denotes $t^+ - t^-$. A transition $t \in T$ is enabled (may fire) at a marking m iff $t^- \leq m$ (i.e. $t^-(p) \leq m(p)$ for every place $p \in P$). When an enabled transition t at a marking m fires, this yields a new marking m' given by $m'(p) := m'(p) + \Delta t(p)$ and denoted by $m \xrightarrow{t} m'$.

Definition 2 (Petri net with time windows in the places). *The pair $\mathcal{P} = (\mathcal{N}, I)$ is called a Petri net with time windows in the places (short: tw-PN) iff*

(i) $\mathcal{N}$ is a (classical) PN and
(ii) $I : P \longrightarrow \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\})$ and for each place $p \in P$ and $I(p) = (l_p, u_p)$ it holds: $l_p \leq u_p$.

The PN $\mathcal{N}$ is called the skeleton of $\mathcal{P}$ and is denoted by $S(\mathcal{P})$. W.l.o.g. we consider the function I with a co-domain $\mathbb{N}_0^+ \times (\mathbb{N}_0^+ \cup \{\infty\})$.

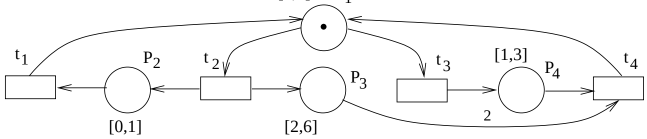


Fig. 1. $\mathcal{P}_1$ is a Petri net with time windows in the places. √

It is obvious that tokens can arrive at a place in different times. Hence, we have to keep the dwell time of each token of every place. This can be solved suprisingly easily using words over numbers. The empty word ε will be assigned to a place without tokens. Each token in a place is presented with a non-negative real number, which is the arriving time of the token in the place (modulo the upper bound of the time interval of the place). We call this kind of presentation of a marking *time marking*.

Definition 3 (time marking). *Let $\mathcal{P}$ be a tw-PN and let P be the set of its places. The map $M : P \longrightarrow \left(\mathbb{R}_0^+\right)^*$ is called a time marking in $\mathcal{P}$.*

By m_M we denote the marking $(l(M(p_1)), l(M(p_2)), \ldots, l(M(p_{|P|})))$. Note that m_M is not a time marking. It is defined by the number of tokens in each place, i.e. m_M is an "usual" marking.

Definition 4 (integer time marking). *A time marking M in $\mathcal{P}$ is called an integer time marking iff $M : P \longrightarrow \mathbb{N}^*$ and P is the set of the places in $\mathcal{P}$.*

Definition 5 (initial time marking). *Let $\mathcal{P}$ be a tw-PN and m_0 be the initial marking in $S(\mathcal{P})$. Then M_0 is the initial time marking on $\mathcal{P}$, iff*

$$M_0(p) := \begin{cases} \varepsilon & , \text{if } m_0(p) = 0 \\ 0^{m_0(p)} & , \text{else} \end{cases} .$$

Obviously, it holds: $m_{M_0} = m_0$ and M_0 is always an integer time marking.

The initial time marking M_0 of $\mathcal{P}_1$ (see Fig. 1) is $M_0 = (0, \varepsilon, \varepsilon, \varepsilon)$.

It is clear that a time marking can change into an other one by firing a transition or by time elapsing. First, we define the notion *ready to fire* and afterwards the notions *change by firing* and *change by time elapsing.*

Definition 6 (ready to fire). *Let M be a time marking such that for each $p \in P$ it holds $M(p) = a_1^p a_2^p \ldots a_{|m(p)|}^p$ and let t be an arbitrary transition in the tw-PN $\mathcal{P}$. Transition t is ready to fire in M, iff*

(i) $t^- \leq m_M$ *(t is enabled in the skeleton $S(\mathcal{P})$),*
(ii) $\forall p (p \in {}^\bullet t \longrightarrow \forall j (j \in \{1, \ldots, t^-(p)\} \longrightarrow l_p \leq a_j^p \leq u_p))$.

Definition 7 (firing a transition). *Let $\mathcal{P}$ be a tw-PN, let T be its set of transitions and let M be an arbitrary time marking in $\mathcal{P}$. The transition $t \in T$ can fire in the time marking M, iff t is ready to fire in M. After the firing, the tw-PN changes into the time-marking M', denoted by $M \xrightarrow{t} M'$, which is defined as follows: Let $M(p) = a_1^p a_2^p \ldots a_n^p$, $t^-(p) = k$ and $t^+(p) = r$[4]. Then it holds:*

$$M'(p) := \begin{cases} a_{k+1}^p \ldots a_n^p 0^r & ,\text{if } k < n \\ 0^r & ,\text{if } k = n \end{cases} .$$

Example 1. Consider $\mathcal{P}_2$ and the time marking $M = (\overbrace{2.5\ 2.0\ 0.7}^{M(p_1)}, \overbrace{1.3}^{M(p_2)}, \overbrace{\varepsilon}^{M(p_3)})$. After firing the transition t_2 the tw-PN $\mathcal{P}_2$ changes into the time marking $M' = (\underbrace{2.0\ 0.7}_{M'(p_1)}, \underbrace{1.3\ 0.0}_{M'(p_2)}, \underbrace{0.0}_{M'(p_3)})$, i.e. $M \xrightarrow{t_2} M'$.

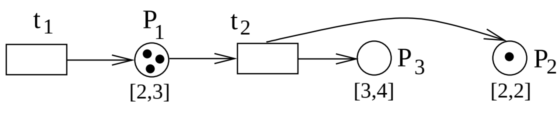


Fig. 2. The tw-PN $\mathcal{P}_2$. √

Remark 1. Let $M_1 \xrightarrow{t} M_2$ be a time marking change in the tw-PN $\mathcal{P}$. Obviously then, t is enabled in the marking m_{M_1} in the skeleton $S(\mathcal{P})$ and it holds: $m_{M_1} \xrightarrow{t} m_{M_2}$. √

[4] Let a be a letter in an alphabet. Then is $a^0 = \varepsilon$.

Definition 8 (time elapsing). *Let $\mathcal{P}$ be a tw-PN and let τ be a non-negative real number. Then the elapsing of time τ in $\mathcal{P}$ is in any time marking always possible. Let M be an arbitrary time marking in $\mathcal{P}$. M is then* **changed** *into the time marking M'* **by the time elapsing** $\tau \in \mathbb{R}_0^+$, *denoted by $M \xrightarrow{\tau} M'$, iff holds:*

Let $M(p) = a_1^p a_2^p \ldots a_n^p$ and let $u_p < a_i + \tau$ and $a_{i+1} + \tau \leq u_p$ be true for each natural number i with $1 \leq i \leq n$. Then the succsessor time marking $M'(p) = b_1^p b_2^p \ldots b_n^p$ is defined as the time marking in $\mathcal{P}$ with

$$b_j^p := \begin{cases} a_{i+j}^p + \tau & ,\text{if } i + j \leq n \\ (a_{i+j-n}^p + \tau)\ \widehat{mod}\ u_p & ,\text{else} \end{cases}.$$

Please note that:

$$a\ \widehat{mod}\ b := \begin{cases} a \bmod b & ,\text{if} \quad a \bmod b \neq 0 \\ b & ,\text{if} \quad a \bmod b = 0 \end{cases}.$$

Example 2. Let $\mathcal{P}$ be a tw-PN and let M be a time marking in $\mathcal{P}$ with

$$M(p) = 3.7\ 2.8\ 2.3\ 2\ 1.5\ 0.3\ 0.1 \text{ and } I(p) = (2, 6).$$

The succsessor time marking M' at the place p after the time elapsing $\tau = 4$ holds then:

$$M'(p) = 6\ 5.5\ 4.3\ 4.1\ 1.7\ 0.8\ 0.3$$

√

The behaviour of a given tw-PN $\mathcal{P} = (P, T, F, V, m_0, I)$ is defined by its changes from a given time marking into another. In general, the changes are an alternating series of time elapsings and firings. Thus, we use the following notions.

A *transition sequence* $\sigma = t_1 t_2 \ldots t_n$ in $\mathcal{P}$ is a word in T^*. A *run* $\sigma(\tau) = \tau_0 t_1 \tau_1 \ldots t_n \tau_n$ is a word in $\mathbb{R}_0^+ (T\, \mathbb{R}_0^+)^*$. The *time-length* $l(\sigma(\tau))$ of the run $\sigma(\tau)$ is the sum $\tau_1 + \ldots + \tau_n$. A run $\sigma(\tau)$ is a *feasible* one in $\mathcal{P}$ if starting in M_0 all time marking changes defined by $\sigma(\tau)$ are possible in $\mathcal{P}$. A transition sequence σ is a *firing sequence* in $\mathcal{P}$ if there exists at least a feasible run $\sigma(\tau)$.

A time marking M is called a *reachable time marking* in $\mathcal{P}$, if there exists a feasable run $\sigma(\tau)$ in $\mathcal{P}$ with $M_0 \xrightarrow{\sigma(\tau)} M$.

The *set of all reachable time markings* in $\mathcal{P}$, starting with time marking M, is denoted by $\mathcal{R}_{\mathcal{P}}(M)$. Obviously, $\mathcal{R}_{\mathcal{P}}(M_0)$ is the set of all reachable time markings in $\mathcal{P}$. Finally, by $\mathcal{R}_{\mathcal{P}}(m_{M_0})$ we denote the set $\{m_M \mid M \in \mathcal{R}_{\mathcal{P}}(M_0)\}$.

A tw-PN $\mathcal{P}$ is *bounded* if the set $\{m_M \mid M \in \mathcal{R}_{\mathcal{P}}(M_0)\}$ is a finite one.

As already stated, time elapsing is always possible in tw-PNs. Moreover, it can occur that in a time marking only time elapsing is possible. This can happen for two different reasons. First, there is no transition enabled in that time marking. Second, a transition is at least enabled but no transition can become ready to fire because of the time restrictions. In the second case, the transition sequence which leads to this time marking can be continued in the skeleton by firing a transition but it cannot be contiued in the tw-PN. We say that all enabled transitions in the considered time marking are in *time-deadlock*. Next definition formalizes this notion.

Definition 9 (time-deadlock). *Let M be a time marking in the tw-PN $\mathcal{P}$. The transition $\hat{t}$ is in M in a time-deadlock (short: t-DL), if*

(i) $\hat{t}^- \leq m_M$

(ii) $\forall \tau (\tau \in \mathbb{R}_0^+ \longrightarrow M \xrightarrow{\tau}\overset{\hat{t}}{\not\rightarrow})$.

Next, we introduce the notion *liveness* for tw-PNs. Actually, there are four levels of liveness. We consider here only the so called 4-liveness, defined by Lautenbach in [11]. This notion will be similarly defined to the definition for the classical PNs.

Definition 10 (liveness). *Let $\mathcal{P}$ be a tP-PN and M a reachable time marking.*

(i) A transition t is live in the time marking M if
$\forall M' (M' \in \mathcal{R}_{\mathcal{P}}(M) \longrightarrow \exists M'' (M'' \in \mathcal{R}_{\mathcal{P}}(M') \land M'' \xrightarrow{t}))$.

(ii) A tw-PN $\mathcal{P}$ is live if all transitions are live in M_0.

3 Reachability and Turing Machines

In this section we compare the sets of reachable markings of a tw-PN and of its skeleton. We will prove that both sets are equal w.r.t. the number of the tokens of the places. Please note that in general there are infinite many time markings with the same number of tokens of the places or, formally said, for each reachable marking m^* in $S(\mathcal{P})$ the set $\{M \mid M \in \mathcal{R}_{\mathcal{P}}(M_0)$ and $m_M = m^*\}$ is infinitely. Afterwards, we show that the power of tw-PNs is not equal to the power of Turing machines, i.e. the tw-PN's are not Turing-complete.

Theorem 1. *Let $\mathcal{P}$ be a tw-PN and $S(\mathcal{P})$ its skeleton. Then an arbitrary firing sequence σ is a firing sequence in $S(\mathcal{P})$ iff σ is a firing sequence in $\mathcal{P}$.*

Idea of the proof:

($\Longleftarrow$) The truth of the sufficiency can be easily proved based on the fact that when a transition can fire in the tw-PN then it is ready to fire. This means that this transition is enabled in the skeleton (see Def. 6).

($\Longrightarrow$) The basic idea for the proof of the necessity is to use the *ultimo rule*, i.e. we wait until the clock of each token reaches a time which is equal to the upper bound of the place where it is situated. Then each enabled transition is also ready to fire. The ultimo rule is realised when between the firing of two transitions always α time units elapse, where $\alpha =$ LCM of all u_p's in $S(\mathcal{P})$, which are natural numbers (not ∞). When a $u_p = \infty$ then the $l_p \neq 0$ is used for the computation of α. If $l_p = 0$ then it will be not considered. For a complete formal proof see [9] or [5]. □

The "waiting time" between the firing of two transitions can of course be reduced. We can always compute a specific "waiting time" $\alpha(t_i, t_j)$ between the firing of the two transitions t_i and t_j which are fired consecutively in the firing sequence. Now, $\alpha(t_i, t_j)$ is computed analogosly to above but it will involve only places which are marked after the firing of t_i and which at least have a post-transition. Please recall that a clock of a token which stays in a place p for more than u_p time units is reset after u_p time units to zero.

The next example illustrates the firing of a transition sequence with the ultimo rule and using specific elapsing times between the firing of two transitions.

Example 3. Let us consider the tw-PN $\mathcal{P}_3$ and the transition sequence $\sigma = t_1t_2t_3$ in the skeleton $S(\mathcal{P}_3)$. This

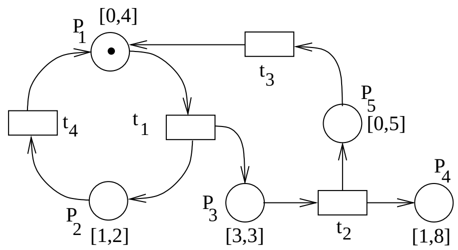


Fig. 3. The tw-PN $\mathcal{P}_3$.

transition sequence is obviously feasible in $S(\mathcal{P}_3)$. In order to show that σ is a firing sequence in $\mathcal{P}_3$ we need to show that there is a run $\sigma(\tau)$ with $M_0 \xrightarrow{\sigma(\tau)}$. According to the "reduced" algorithm for firing due to the ultimo rule the run we search for should start with the time elapsing $\tau_0 = 4$. This is because the only marked place in M_0 is P_1 with $u_{P_1} = 4$. Then t_1 fires. In the successor time marking the marked places are P_2 and P_3 and both have post-transitions. Thus the elapsing time $\tau_1 = LCM\{2, 3\} = 6$. Then t_2 can fire. Now, although the marked places in the followed time marking are P_2, P_4 and P_5, only P_2 and P_5 are relevant for computing τ_2 because P_4 has no post-transitions. Eventually, $\tau_2 = 10$ and subsequently t_3 can fire. Thus, $\sigma(\tau) = 4\ t_1\ 6\ t_2\ 10\ t_3$ is a feasible run in $\mathcal{P}_3$. √

The next theorem follows immediately from Theorem 1.

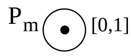
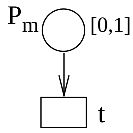
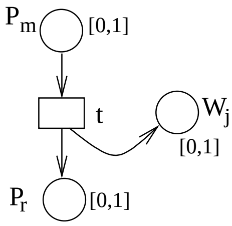
Theorem 2. *The sets $\mathcal{R}_{\mathcal{P}}(m_{M_0})$ of all reachable markings (not considered as time markings) of an arbitrary tw-PN $\mathcal{P}$ is equal to the set $\mathcal{R}_{S(\mathcal{P})}$ of all reachable markings of its skeleton.*

Please note, that although both sets of reachable markings of a tw-PN and of its skeleton are equal, it is possible to reach a time marking in the tw-PN such that all enabled transitions are in a t-DL in them. In Section 5 we state some examples for this fact.

In the second part of this section we show that the power of tw-PNs is not equal to the power of Turing machines. For this purpose we prove the non-equivalence between the power of tw-PNs and counter machines. As it is well known, each Turing machine can be simulated by a counter machine and vice versa (cf. [4]).

A counter machine consists of a finite numbers of counters $K_1, \dots K_n$ and a numbered program comprising the 4 different commands START, HALT, INC and DEC.

The next table recalls the syntax and the semantics of the four commands and shows how the START, HALT and INC, can be simulated using a tw-PN. We will prove that the command DEC cannot be simulated.

Counter machine command	Description of the command	Model of the numbered command as a tw-PN
0 : START : m	Start the program and go to line m.	
m : HALT	Stop the program.	
m : INC(j) : r	Increment counter j by 1 then go to line r.	
m : DEC(j) : r : s	If counter j equals 0 then go to line r, else decrement counter j and go to line s.	-

The reason for the impossibility to simulate the command DEC with a tw-PN is because it is impossible to model the *zero-test* with them. The zero-test for a (tw-)PN is the check whether a place is marked or not. More exactly, the zero-test is the following. Let us consider three places p, p_{marked} and p_{empty} in a (tw-)PN. In the initial marking, the places p_{marked} and p_{empty} are not marked. If p is marked in the initial marking, then there is a run and after its firing the place p_{marked} is marked, the place p_{empty} is not marked and there is no run that can change this. If p is not marked in the initial marking, then there is a run and after its firing the place p_{empty} is marked, the place p_{marked} is not marked and there is no run that can change this.

Assuming that the power equivalence between tw-PNs and counter machines holds, this leads to the power equivalence between classical PNs and counter machines. The reason for this is the fact that each firing sequence in the skeleton is also a firing sequence in the tw-PN. That means there is a feasible run of each transition sequence which can be fired in the skeleton (cf. Theorem 1). Additionally, in the tw-PN there are feasible runs which cannot be continued with an enabled transition because the transition is in a t-DL in the current time marking. Consequently, for a transition sequence $\sigma = \sigma_1 t_k$ there is a feasible run $\sigma_1(\tau)$ with $M_0 \xrightarrow{\sigma_1(\tau)} M$ and t_k is enabled in M but also in t-DL in M. Then, σ is a firing transition sequence in the skeleton and with it a firing sequence in the tw-PN (with another feasible run), too. Thus, a zero-test for a place p with a tw-PN is a zero-test (same firing sequences) for p with a classical PN. That leads immediately to the power equivalence between classical PNs and Turing machines. The last is obviously a contradiction.

In summary we can state the following theorem.

Theorem 3. *The power of tw-PNs is not equal to the power of Turing machines.*

Lastly, the power of the tw-PNs is bounded below. This follows from the next theorem.

Theorem 4. *The power of tw-PNs is not less than to the power of the classical PNs.*

Proof. Each classical PN $\mathcal{N}$ can be presented as a tw-PN with $l_p := 0$ and $u_p := \infty$ for each place p in $\mathcal{N}$. With it the proposition follows immediately. □

4 Feasible Runs in a tw-PN

In this section we consider feasible runs in tw-PNs and compare these with runs in Time Petri Nets (TPN).

The TPNs are defined by Merlin in [15] and well studied in [2], [6] and [8]. These time dependent PNs are derived from classical Petri nets, where each transition t is associated with a time interval $[a_t, b_t]$. Here a_t and b_t are relative to the time, when t was last enabled. When t becomes enabled, it cannot fire before a_t time units have elapsed and it has to fire not later than b_t time units, unless t got disabled in between by the firing of another transition. The firing itself of a transition takes no time. The time interval is described by real numbers, but the interval bounds are non-negative rational numbers. It is easy to see (cf. [6]) that w.l.o.g. the interval bounds can be considered as integers only. Thus, the interval bounds a_t and b_t of any transition t are natural numbers, including zero and $a_t \le b_t$ or $b_t = \infty$. It is clear that in general a transition sequence can have infinitely many feasible runs. However, if a feasible run of a transition sequence takes α time units and an other run of the same sequence takes β time units ($\alpha < \beta$), then to each $\gamma \in [\alpha, \beta]$ there is also a feasible run of the sequence which time duration is exactly γ time units. The behaviour of the feasible runs in a tw-PN is quite different.

In a tw-PN it is possible to find a transition sequence with "time gaps". This means that the time continuum $[0, \infty]$ can be divided in intervals $[0, a_0], [a_0, a_1], [a_1, a_2], [a_2, a_3], [a_3, a_4], \dots$ and it is possible to find a feasible run $\sigma(\tau^i)$ of the transition sequence σ with a time-length $l(\sigma(\tau^i)) \in [a_i, a_{i+1}]$ for $i = 0, 2, 4, \dots$ but there is no feasible run for which time-length is a number in $[0, a_0]$ or in $[a_i, a_{i+1}]$ for $i = 1, 3, 5, \dots$.

Theorem 5. *Let $\mathcal{Z}$ be a TPN and $\sigma = t_1 t_2 \dots t_n$ an arbitrary transition sequence in $\mathcal{Z}$. Furthermore let $\sigma(\tau_\alpha) = \tau_0^\alpha\ t_1\ \tau_1^\alpha\ t_2\ \tau_2^\alpha \dots \tau_{n-1}^\alpha\ t_n\ \tau_n^\alpha$ and $\sigma(\tau_\beta) = \tau_0^\beta\ t_1\ \tau_1^\beta\ t_2\ \tau_2^\beta \dots \tau_{n-1}^\beta\ t_n\ \tau_n^\beta$ be two feasible runs of the transition sequence σ, with $l(\sigma(\tau_\alpha)) = \alpha$ and $l(\sigma(\tau_\beta)) = \beta$, $\alpha < \beta$. Then for each $\gamma \in [\alpha, \beta]$ there exists a feasible run $\sigma(\tau_\gamma) = \tau_0^\gamma\ t_1\ \tau_1^\gamma\ t_2\ \ \tau_2^\gamma \dots \tau_{n-1}^\gamma\ t_n\ \tau_n^\gamma$ with $l(\sigma(\tau_\gamma)) = \gamma$.*

Idea of the proof. A feasible run can be considered as a concrete instance of a "parametric run". A parametric run is a run which times are not numbers but variables (parameters) satisfying some restrictions. For each transition sequence there is one parametric run, e.g. for the sequence $\sigma = t_1 t_2 \dots t_n$ the parametric run is $\sigma(X) = x_0 t_1 x_1 t_2 x_2 \dots x_{n-1} t_n x_n$. The variables have to fulfill certain constrains derived from the time intervals of the transitions belonging to the sequence σ. These are described by linear inequalities and define a polyhedron $H_\sigma \subseteq \mathbb{R}_0^+$. A formal and complete definition of a parametric run can be found in [10].
Now, the time-length of the parametric run of the transition sequence σ is given by the linear function

$$f(X) := \sum_{i=0}^{n} x_i \quad \text{and} \quad X = (x_o, \dots, x_n) \text{ satisfy } H_\sigma.$$

The time-length of $\sigma(\tau_\alpha)$ is then $f(X) = \alpha$ for $X = \tau_\alpha$ and the time-length of $\sigma(\tau_\beta)$ is then $f(X) = \beta$ for $X = \tau_\beta$. Obviously the values α and β are real numbers between $\mu = \min\{f(X) \mid X \text{ satisfied } H_\sigma\}$ and $\lambda = \max\{f(X) \mid X \text{ satisfied } H_\sigma\}$[1].

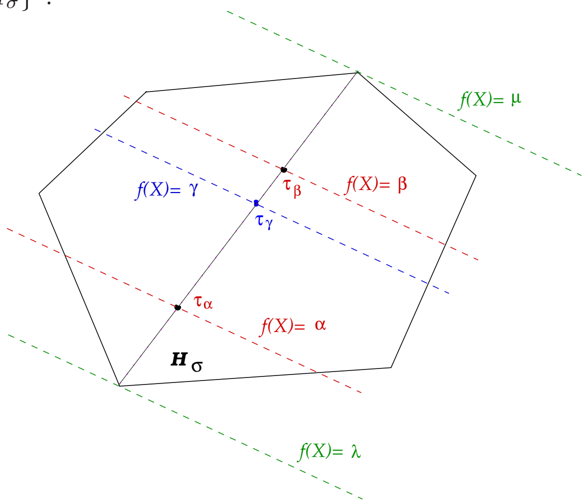


Fig. 4. Graphical ilustration of H_σ for $n = 2$.

Because the polyhedron is convex and the points τ_α and τ_β belong to (satisfied) H_σ, there is a point τ_γ, which belongs to H_σ (to the segment, defined by the points τ_α and τ_β), and lies on a hyperplane, parallel to the hyperplanes $f(X) = \alpha$ and $f(X) = \beta$ (cf. [13]). This means that the function $f(X) = \gamma$ for $X = \tau_\gamma$. □

This run property is not true for a tw-PN in general. It is possible that there are "gaps" in the time length possibilities for the runs of a firing sequence. That means it is possible to find a tw-PN $\mathcal{P}$ and a firing sequence σ with two feasible runs whose time lengths are α and β in $\mathcal{P}$ but there does not exist a run of σ with a length γ, $\alpha < \gamma < \beta$. Of course, this is neither true for all α and β nor for all firing sequences.

The next example verifies this fact.

Example 4. Let us consider the tw-PN $\mathcal{P}_4$ given in Fig. 5 and the transition sequence $\sigma = t_1t_2t_3$. The runs $3\,t_1 3\,t_2 3\,t_3$ and $5\,t_1 2\,t_2 3\,t_3$ are feasible runs of σ_1 with the time lengths 9 and 10. It is easy to see that there does not exist a run of σ_1 whose time length is e.g. 9.5 or any one other number between 9 and 10. However, the lengths of all feasible runs of σ belong to the intervals $[7, 9], [10, 12], [13, 15], \ldots$.

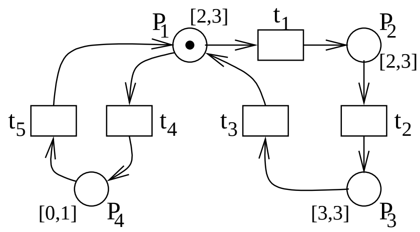


Fig. 5. The tw-PN $\mathcal{P}_4$. √

Considering the sequence $\sigma_2 = t_4t_5t_4$, it is obvious that the shortest time length of a feasable run of σ_2 is 4. Furthermore, it can be easily seen that for each real number $k, k \geq 4$ a feasible run of σ_2 exists whose time length is k.

[1] If λ exists. Otherwise α and β are not bounded above.

At the end of this section we compare the behaviour of a tw-PN when the time elapsings are real numbers with its behaviour when the time elapsings are natural numbers. We did this comparison for TPNs (cf. [10]) and proved that for these time dependent PNs (at least) the reachability behaviour and the liveness behaviour are the same. This is not true for tw-PNs in general. Fig. 6 illustrates this fact.

Example 5. Here we use the notion "integer reachability graph" for a tw-PN which is

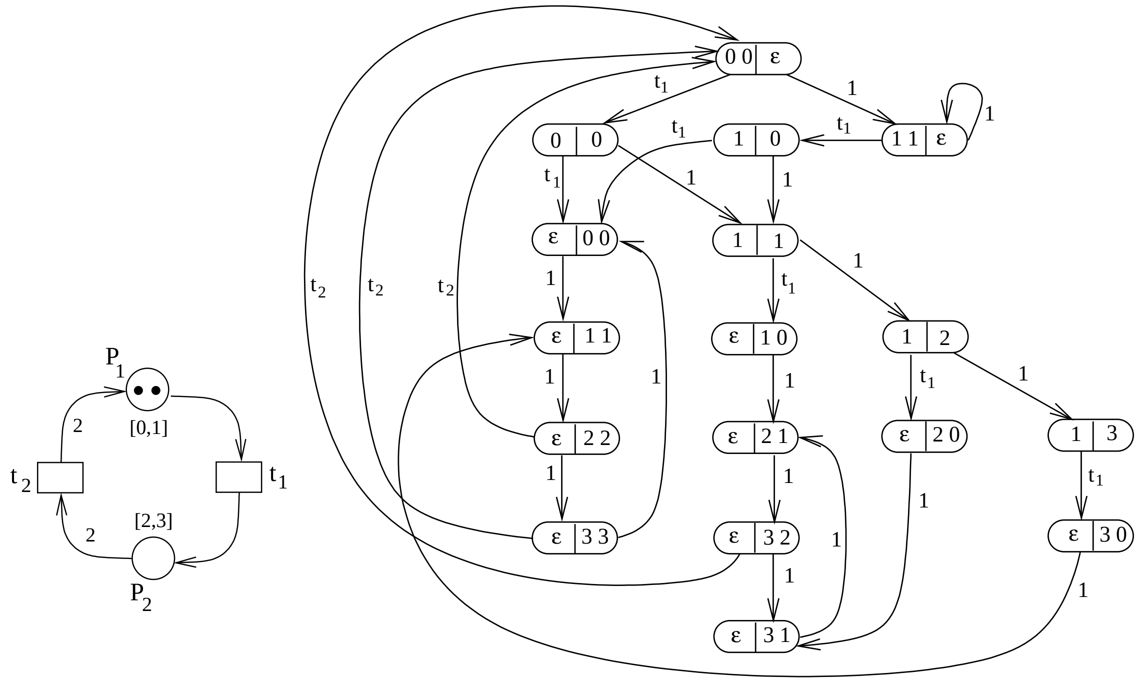


Fig. 6. The tw-PN $\mathcal{P}_5$ and its integer reachability graph.

a reachability graph (labeled directed graph) obtained considering all reachable integer time markings as its set of vertices. The edges are defined by the triples (M, t, M') and $(M, 1, M')$, where $M \xrightarrow{t} M'$ and $M \xrightarrow{1} M'$, respectively and M and M' are time markings, t is a transition, and 1 stands for one time unit. This reachability graph can, of course, be reduced. A formal definition can be found in the Appendix.

In the reachability graph from above each vertex is divided into two parts. Each of them contain a word over $\mathbb{N}^*$ which describes the dwell time of each token in the place p (modulo u_p). Thus, the text $\varepsilon \mid 3\ 2$ in a vertex models the time marking M with $M(p_1) = \varepsilon$ (p_1 is not marked) and $M(p_2) = 3\ 2$ (p_2 has been contained 2 tokens for 3 resp. 2 time units modulo $3 = u_{p_2}$).

As can easily be seen, the run $\sigma(\tau^r) = 1.5\ t_1\ 1.5\ t_1$ is a feasable one in the tw-PN $\mathcal{P}_5$. After executing this run $\mathcal{P}_4$ is in the time marking $M^r = (\varepsilon, 1.5\ 0)$ and thus t_2 is enabled, but it is in t-DL. Subsequently, M^r is a leaf in the "real" reachability graph of $\mathcal{P}_5$. However, the "integer" reachability graph does not contain any leafs and, therefore, the "real" and the "integer" behaviour of P_5 are different. $\surd$

Finally, we give a necessary condition for having a difference between the "real" and the "integer" behaviour of a tw-PN. The proof is given in the Appendix.

Theorem 6. *Let $\mathcal{P}$ be a tw-PN and let $t \in T$ be a transition so that $^\bullet t = \{p\}$. Furthermore, let $u_p = l_p \cdot V(p,t) - 1$.*

Then the following is true:

(i) a time marking M^r exists, so that $|M^r(p)| = V(p,t)$ and t is in M^r in a t-DL,
(ii) no time marking M^{int} exists, so that $M^{int}(p) \in \mathbb{N}^*$, $|M^{int}(p)| = V(p,t)$ and t is in M^{int} in a t-DL.

Obviously, the place p_2 in the tw-PN $\mathcal{P}_5$ (Fig. 6) satisfy the condition $u_p = l_p \cdot V(p,t) - 1$.

5 Liveness

In this section we compare the liveness behaviour of tw-PN with the liveness behaviour of its skeleton and show that both are not equal in general. Eventually, we give a result for a class of restricted tw-PNs whose liveness behaviour is equal to that of its skeleton.

Remark 2. When a tw-PN $\mathcal{P}$ is live then $S(\mathcal{P})$ is live as well. In general the opposite does not hold.

Proof. It is clear that when an arbitrary run of a sequence σ can be executed in an arbitrary tw-PN, then the sequence can be fired in the skeleton of the net. It follows that when a tw-PN is live, its skeleton is live as well. The opposite can be quickly falsified with an example. Let us consider the tw-PN $\mathcal{P}_6$ given in Fig. 7. It is obvious that S($\mathcal{P}_6$) is live. Moreover, after executing the feasable run t_1 5.0 t_1 in $\mathcal{P}_6$, it is easily seen that t_2 is the only feasable transition but it cannot become ready to fire, i.e. t_2 is in t-DL in this time marking. Therefore, $\mathcal{P}_6$ is not live.

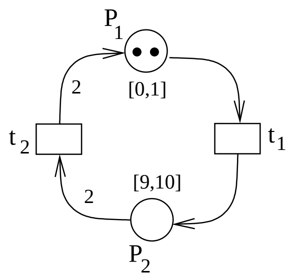


Fig. 7. The tw-PN $\mathcal{P}_6$.

□

However, there is a structural restricted class of tw-PNs whose liveness behaviour is the same as that of its skeleton. The next theorem introduces this class.

Theorem 7. *Let $\mathcal{P}$ be a tw-PN and let $S(\mathcal{P})$ be its skeleton so that $|^\bullet t| \leq 1$ for every $t \in T$ and so that $S(\mathcal{P})$ is live. Furthermore, let the following estimate hold for all places $p \in P$:*

$$l_p \leq \frac{u_p}{\max\limits_{t \in T} V(p,t)}. \quad (1)$$

Then $\mathcal{P}$ is live.

The proof is directly derived from Theorem 8 (see Appendix).

Please note that the restriction $|^\bullet t| \leq 1$ is an essential one. The tw-PN $\mathcal{P}_7$ shown in Fig. 8 gives an example for a net whose skeleton $S(\mathcal{P}_7)$ is live, $V(f) = 1$ for all $f \in F$ but $\mathcal{P}_7$ is not live. This fact can be verified considering the run $\sigma(\tau) := 2\, t_1$. It is clear that t_2 is in a t-DL in the time marking M (with $M_0 \xrightarrow{\tau(\sigma)} M$) and, therefore, $\mathcal{P}_7$ is not live.

Please also note that Theorem 7 gives a sufficient condition only. By doing a small modification to the tw-PN $\mathcal{P}_6$ we gain a tw-PN $\mathcal{P}_8$ that violates condition (1) in Theorem 7, but it is still live.

Corollary 1. *Let $\mathcal{P}$ be a tw-PN with $V(f) = 1$ for each $f \in F$ and $|^\bullet t| \leq 1$. Then $\mathcal{P}$ is live iff $S(\mathcal{P})$ is live.*

Corollary 2. *Let $\mathcal{P}$ be a tw-PN and let $S(\mathcal{P})$ be live. Let $|^\bullet t| \leq 1$ for every $t \in T$. Furthermore $l_p = 0$ or $u_p = \infty$ is true for each place p in $\mathcal{P}$. Then $\mathcal{P}$ is live, too.*

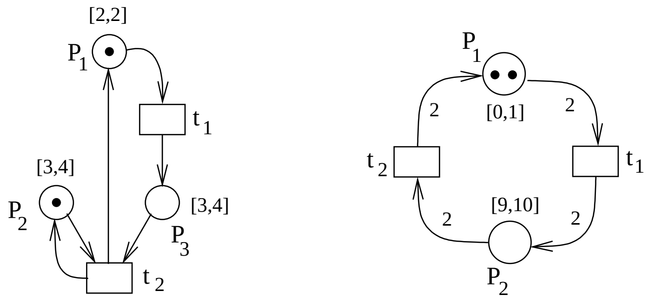


Fig. 8. Left: The tw-PN $\mathcal{P}_7$. Right: The tw-PN $\mathcal{P}_8$.

6 Conclusion

In this paper we have studied PNs with explicit time restrictions (windows) added to the places, the tw-PNs. More precisely, a time interval $[l_p, u_p]$ is assigned to each place p. Each token, which arrives at the place p cannot leave it before it has spent at least l_p time units there. During the time l_p until u_p, which begins upon the tokens arrival, the place opens its "window" for *this* token and *this* token can leave. Each token counts the time with "its own clock". After u_p time units if the token, for whatever reason, has not left the place, its clock resets, and the procedure is repeated.

Usually time dependent PNs are equivalent to Turing machines. Here we have shown, however, that this does not hold for the power of the tw-PNs. Moreover, each classical PN can be considered as a tw-PN and thus the power of the tw-PNs is not less than that of the classical PNs.

Another untypical result for time dependent PNs is that the set of reachable markings (considering the number of tokens of the places only) is not decreased in comparison with the set of the same tw-PN without the time (skeleton of the tw-PN). The reason for this fact is that a force for the transitions to fire is absent in tw-PNs. However, the reachability behaviour, which means not only the set of all reachable (time) markings, but also includes the set of all possible paths between two *time* markings and the liveness behaviour of a tw-PN, generally differ from the reachability behaviour and the liveness behaviour of its skeleton.

For a structural restricted class of tw-PNs, we have shown that the liveness behaviour can be the same as that of its skeleton. This means the time has no influence on the liveness behaviour. In future research, we will pursue further classes of tw-PNs with time-invariant reachability and liveness behaviors.

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7 Appendix

Definition 11 (integer reachability graph). *Let $\mathcal{P}$ be a tw-PN with the skeleton $S(\mathcal{P}) = (P, T, F, V, m_0)$ and let $IM(\mathcal{P})$ be the set of all reachable integer time markings in $\mathcal{P}$. The integer reachability graph of $\mathcal{P}$ is the directed graph $\mathcal{IG}_{\mathcal{P}} = (IM(\mathcal{P}), E)$ with*

Basis*:*
$IM(\mathcal{P}) := \{M_0\}; E := \emptyset$

Step*:*
Let $M \in IM(\mathcal{P})$ already. Then

1. *for each $t \in T$*
   if $M \xrightarrow{t} M'$ possible in $\mathcal{P}$ then $IM(\mathcal{P}) := IM(\mathcal{P}) \cup \{M'\}$;
   $E := E \cup \{(M, t, M')\}$
   end if
2. *if $M \xrightarrow{1} M'$ possible in $\mathcal{P}$ then $IM(\mathcal{P}) := IM(\mathcal{P}) \cup \{M'\}$;*
   $E := E \cup \{(M, 1, M')\}$
   end if.

Definition 12 (equidistant time marking). *Let $\mathcal{P}$ be a tw-PN. A time marking M is called an equidistant time marking in the place p if $M(p) = a_1^p, \ldots, a_n^p$ with*

$$a_{j+1}^p - a_j^p = \frac{u_p}{|M(p)|} \text{ for } j = 1, \ldots, n-1 \quad \text{and} \quad (u_p - a_n^p) + a_1^p = \frac{u_p}{|M(p)|}.$$

Theorem 8. *Let $\mathcal{P}$ be a tw-PN and let t be a transition with ${}^\bullet t = \{p\}$.*
For each time markings M, m^* and M^e where $|M^*(p)| = |M^e(p)| = |M(p)|$ the following statements are equivalent:

(i) *There is a time marking M^* so that t is in a time-DL in M^*.*
(ii) *t is in time-DL in the equidistant time marking M^e.*
(iii) $l_p > u_p \left(1 - \frac{V(p,t)-1}{|M(p)|}\right)$.

Proof. **This proof** is published in [5].
We set $n := |M(p)| = |M^*(p)| = |M^e(p)|, \lambda := V(p, t)$.
$(ii) \Longrightarrow (i)$ trivial
$(iii) \Longrightarrow (ii)$ We have

$$\begin{aligned} a_1 - a_\lambda &= (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{\lambda-1} - a_\lambda) \\ &= (\lambda - 1)\frac{u_p}{n} \\ &= u_p - u_p + (\lambda - 1)\frac{u_p}{n} \\ &= u_p - u_p \left(1 - \frac{\lambda - 1}{n}\right) \\ &> u_p - l_p \end{aligned}$$

and subsequently t is in M^e in a time-DL.

$(i) \implies (iii)$ W.l.o.g. we can assume that $l_p, u_p \in \mathbb{N}$ and $a_n^p = u_p$ for the time marking M^*.

For an easier formal writing we copy the interval $[0, u_p]$ and add it to the end, i.e. we consider the time marking $\widetilde{M}^*(p) := \widetilde{a}_1^p, \ldots, \widetilde{a}_n^p, \widetilde{a}_1^p + u_p, \ldots, \widetilde{a}_n^p + u_p$.

Now let $\chi_{j,k}$ be the number of tokens in the interval $[j, k]$.

Furthermore there exists a natural number $\kappa := |\{a^p \in \widetilde{M}^*(p) : a \in \mathbb{N}\}|$. We assumed that $a_n^p = u_p$ thus $\kappa \geq 1$.

Therefore we have

$$\begin{aligned} n &= \sum_{j=0}^{u_p-1} \chi_{j,j+1} - \kappa && (2) \\ &= \frac{\sum_{j=0}^{u_p-1} \chi_{j,u_p-l_p+j} - \kappa}{u_p - l_p} \\ &\leq \frac{\sum_{j=0}^{u_p-1} (\lambda - 1) - \kappa}{u_p - l_p} \\ &< \frac{\sum_{j=0}^{u_p-1} (\lambda - 1)}{u_p - l_p} && (3) \\ &= \frac{u_p(\lambda - 1)}{u_p - l_p} \end{aligned}$$

where equality (2) holds as κ tokens are counted twice and inequality (3) holds as $\kappa \geq 1$.

And thus

$$\begin{aligned} & n < \frac{u_p}{u_p - l_p}(\lambda - 1) \\ \implies & u_p - l_p < \frac{u_p}{n}(\lambda - 1) \\ \implies & l_p > u_p - u_p\left(\frac{\lambda - 1}{n}\right) = u_p\left(1 - \frac{\lambda - 1}{n}\right). \end{aligned}$$

□

Proof of **Theorem 6**. It obviously implicates from $u_p = l_p \cdot V(p,t) - 1$ that

$$\lambda := V(p,t) \geq 2, \tag{4}$$

otherwise it would follow that $u_p = l_p - 1$ and thus $u_p < l_p$.

(i) Let M^r be a time marking so that M^r is an equidistant time marking in the place p and so that no other place is marked, i.e. $M^r(p) = a_1 \ldots a_\lambda$ and $a_k - a_{k+1} = \frac{u_p}{\lambda}$, $k = 1, \ldots, \lambda - 1$, and $M^r(\widetilde{p}) = \varepsilon$ for all $\widetilde{p} \neq p$.

We will show that t is in a t-DL in M^r. According to the conditions $\lambda = |M^r(p)|$ and $u_p = l_p \cdot \lambda - 1$ the following holds

$$\begin{aligned} u_p\left(1 - \frac{\lambda - 1}{\lambda}\right) &= \frac{u_p}{\lambda} \\ &= \frac{l_p \cdot \lambda - 1}{\lambda} \\ &= l_p - \frac{1}{\lambda} \\ &< l_p. \end{aligned}$$

Thus we have

$$l_p > u_p \left(1 - \frac{\lambda - 1}{|M^r(p)|}\right)$$

and from Theorem 8 it follows that t is in M^r in a t-DL.

(ii) This part is proven in two steps.

First we show that for any time marking M so that t is in M in a t-DL, all "time distances" between two tokens must be less than l_p. Second we show that for any time marking M with $M(p) \in \mathbb{N}^*$ there is at least one "distance" greater (or equal) than l_p.

Now let M be a time marking such that $|M(p)| = \lambda$ and t is in M in a t-DL. Furthermore let $M(p) := a_1 \dots a_\lambda$. Assume that there are two tokens such that the "distance" of their times is greater or equal than the lower bound of the place, formally speaking: assume that a natural number $k \in \{1, \dots, \lambda\}$ exists so that $a_k - a_{k+1} \geq l_p$, if $k \neq \lambda$, and $u_p - a_1 + a_\lambda \geq l_p$, if $k = \lambda$. W.l.o.g. we can assume that $k = \lambda$, i.e. $u_p - a_1 + a_\lambda \geq l_p$, and $a_1 = u_p$. Then it follows

$$l_p \leq u_p - a_1 + a_\lambda = u_p - u_p + a_\lambda = a_\lambda.$$

This means that the dwell times of all tokens are in the intervall $[l_p, u_p]$ and thus the transition t is ready to fire. But this is a contradiction to the condition that t is in M in a t-DL.

Now we show the second step. We still need to prove that no time marking M exists so that $M(p) \in \mathbb{N}^*$ and t is in M in a t-DL. For this purpose let us assume that M^{int} is a time marking, with $M^{int}(p) = b_1 \dots b_\lambda$, $b_k \in \mathbb{N}$ for each $k \in \{1, \dots, \lambda\}$ and t is in M^{int} in a t-DL. Please note, that M^{int} does not need to be a reachable one in $\mathcal{P}$.

From the first step it follows that $b_k - b_{k+1} < l_p$ for all $k = 1, \dots, \lambda - 1$ and $u_p - b_1 + b_\lambda < l_p$. Therefore it holds

$$b_1 - b_\lambda = \sum_{k=1}^{\lambda-1} b_k - b_{k+1} < \sum_{k=1}^{\lambda-1} l_p = (\lambda - 1) \cdot l_p.$$

It follows from $l_p \in \mathbb{N}$ and by the condition $M^{int}(p) \in \mathbb{N}$ that

$$b_1 - b_\lambda \leq (\lambda - 1) \cdot (l_p - 1) = \lambda \cdot l_p - \lambda - l_p + 1 \tag{5}$$

holds.

Finally, w.l.o.g. we can assume that $b_1 = u_p$. Then it follows from $u_p = l_p \cdot \lambda - 1$ and from (5) that

$$\begin{aligned} b_\lambda &= u_p - (b_1 - b_\lambda) \\ &\geq u_p - \lambda \cdot l_p + \lambda + l_p - 1 \\ &= \lambda \cdot l_p - 1 - \lambda \cdot l_p + \lambda + l_p - 1 \\ &= l_p + \lambda - 2. \end{aligned}$$

Subsequently and because of (4) it follows immediately that $b_\lambda \geq l_p$. This contradicts the assumption that t is in M^{int} in a t-DL. □