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Petri Nets with Time Windows: A Comparison to Classical Petri Nets

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Abstract. We present Petri nets with time windows (tw-PN) where each place is associated with an interval (window). Every token which arrives at a place gets a real-valued clock which shows its "age". A transition can fire when all needed tokens are "old enough". When a token reaches an "age" equal to the upper bound of the place where it is situated, the "token's age", i.e., clock will be reset to zero. Following this we compare these time dependent Petri nets with their (timeless) skeletons. The sets of both their reachable markings are equal, their liveness behaviour is different, and neither is equivalent to Turing machines. We also prove the existence of runs where time gaps are possible in the tw-PN, which is an extraordinary feature.

1. Introduction

Petri nets with time windows (tw-PN) are derived from classical Petri nets (PN) where each place p is associated with a time interval $[l_p, u_p]$. When a token arrives in a place p, it can not leave p before l_p time units have elapsed. During the time interval (window) $[l_p, u_p]$ the token can leave p. At the end of the interval there is not a force for leaving. When the token remains longer in the place p as u_p time units then the current time of the token in the place p is reset modulo u_p. When t becomes enabled, it can fire

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when enough tokens in its input places can leave them. In other words, t can fire if t is enabled and all time windows of enough tokens in all input places of t are “open”. The firing itself of a transition takes no time. The time is represented by non-negative real numbers, but the interval bounds are non-negative rational numbers. It is easy to see that w.l.o.g. the interval bounds can be considered as integers only. Thus, the interval bounds l_p and u_p of any place p are natural numbers, including zero and $l_p \leq u_p$ or $u_p = \infty$.

Every possible situation in a given tw-PN can be described completely by a time marking M with $M(p) \in (\mathbb{R}_0^+)^*$ for each place p. Thus, a time marking is a vector of words over $\mathbb{R}_0^+$. In general, each tw-PN has (i.e. the state space of the tw-PN contains) infinite number of time markings.

The tw-PN was first introduced in [8] and later applied for modelling and diagnosis in the automation engineering in [5].

Related work: In the classical Petri nets the time is only implicitly involved in the kind of causal context. The works of Merlin [7] and Ramchandani [11] certainly started a new branch of Petri nets - the time dependent Petri nets. Merlin defined the Time Petri nets (TPN) and Ramchandani the Timed Petri nets. Since this time a huge amount of different kinds of time associations have been defined. Time can be added to transitions ([7], [11], [6]), to places ([12], [8], [5]) and to edges ([2], [1]) in various ways. Classes of well known time dependent Petri nets are given in [13]. However, most of them are equivalent to the Turing machines and thus the most interesting problems like the reachability problem and the liveness problem are undecidable.

The paper is organised as follows: The next section gives some preliminary definitions and remarks. The third section compares the reachability behaviour of an arbitrary tw-PN with its skeleton. Afterwards, the non-equivalence between tw-PNs and Turing machines is proved. The fourth section deals with liveness behaviour of a tw-PN and its skeleton. Finally, the last section summerizes the results and gives a remark including future outlook.

2. Basic Notations and Definitions

As usual, we use the following notations in this paper: $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^+ := \mathbb{N}\backslash\{0\}$. $\mathbb{Q}_0^+$ is the set of non-negative rational numbers and $\mathbb{R}_0^+$ the set of non-negative real numbers. By $|A|$ we denote the number of elements of a finite set A. T^* denotes the language of all words over the alphabet T, including the empty word ε, the natural number $l(\omega)$ is the length of ω. The word a^k with $a \in T$ and $k \in \mathbb{N}$ stands for the word $\underbrace{a \ldots a}_{k-times}$.

Definition 2.1. (classical Petri net)
The structure $\mathcal{N} = (P, T, F, V, m_o)$ is called a (classical) Petri net (short: PN) iff

(i) P, T, F are finite sets with
$P \cap T = \emptyset$, $P \cup T \neq \emptyset$, $F \subseteq (P \times T) \cup (T \times P)$ and $dom(F) \cup cod(F) = P \cup T$

(ii) $V : F \longrightarrow \mathbb{N}^+$ (weight of the arcs)

(iii) $m_o : P \longrightarrow \mathbb{N}$ (initial marking)

A marking of a PN is a (total) function $m : P \longrightarrow \mathbb{N}$, such that $m(p)$ denotes the number of tokens at the place p. The pre-sets and post-sets of a transition t are given by $^\bullet t := \{p \mid p \in P \wedge (p,t) \in F\}$ and $t^\bullet := \{p \mid p \in P \wedge (t,p) \in F\}$, respectively. Each transition $t \in T$ defines the markings t^- and t^+ as follows:

$$t^-(p) = \begin{cases} V(p,t) & \text{iff} \quad (p,t) \in F \\ 0 & \text{iff} \quad (p,t) \notin F \end{cases} \qquad t^+(p) = \begin{cases} V(t,p) & \text{iff} \quad (t,p) \in F \\ 0 & \text{iff} \quad (t,p) \notin F \end{cases}$$

Moreover, Δt denotes $t^+ - t^-$. A transition $t \in T$ is enabled (may fire) at a marking m iff $t^- \leq m$ (i.e. $t^-(p) \leq m(p)$ for every place $p \in P$). When an enabled transition t at a marking m fires, this yields a new marking m' given by $m'(p) := m(p) + \Delta t(p)$ and is denoted by $m \xrightarrow{t} m'$.

Definition 2.2. (Petri net with time windows in the places)
The pair $\mathcal{P} = (\mathcal{N}, I)$ is called a Petri net with time windows in the places (short: tw-PN) iff

(i) $\mathcal{N}$ is a (classical) PN and

(ii) $I : P \longrightarrow \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\})$ and for each place $p \in P$ and $I(p) = (l_p, u_p)$ it holds: $l_p \leq u_p$.

The PN $\mathcal{N}$ is called the skeleton of $\mathcal{P}$ and is it is denoted by $S(\mathcal{P})$. W.l.o.g. we consider the function I with a co-domain $\mathbb{N}_0^+ \times (\mathbb{N}_0^+ \cup \{\infty\})$.

Example 1.

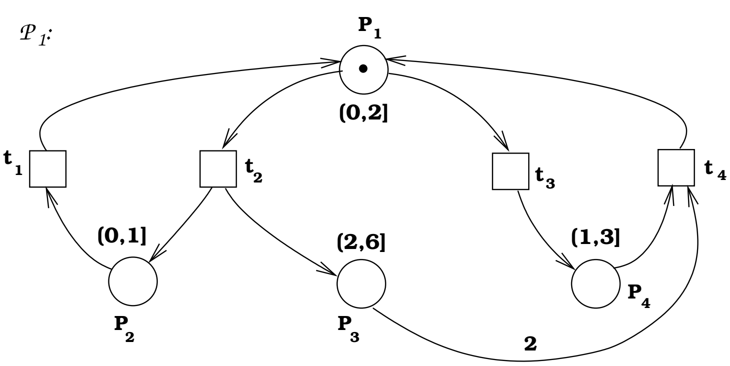


Figure 1. $\mathcal{P}_1$ is a Petri net with time windows in the places.

It is obvious that tokens can arrive at a place in different times. Hence, we have to keep the arriving time of each token of every place. This can be solved suprisingly easily using words over numbers. The empty word ε will be assigned to a place without tokens. Each token in a place is presented with a non-negative real number, which is the arriving time of the token in the place (modulo the upper bound of the time interval of the place). We call this kind of presentation of a marking *time marking*.

Definition 2.3. (time marking)
Let $\mathcal{P}$ be a Petri net with time windows in the places. The map $M : P \longrightarrow \left(\mathbb{R}_0^+\right)^*$ is called a time marking in $\mathcal{P}$.

By m_M we denote the usual (not time) marking

$$(l(M(p_1)), l(M(p_2)), \ldots, l(M(p_{|P|})))$$

which corresponds to the time marking M: the word $M(p_i), i = 1 \ldots |P|$ describes the duration of dwell of the tokens in the place p_i (modulo u_{p_i}) and the length $l(M(p_i))$ of the word $M(p_i)$ is equal to the number of tokens in the place p_i at the time marking M.

Definition 2.4. (initial time marking)
Let $\mathcal{P}$ be a tw-PN and m_0 be the initial marking in $S(\mathcal{P})$. Then M_0 is the initial time marking on $\mathcal{P}$, iff

$$M_0(p) := \begin{cases} \varepsilon & \text{if } m_0(p) = 0 \\ 0^{m_0(p)} & \text{else} \end{cases} .$$

Obviously, it holds: $m_{M_0} = m_0$.

The initial time marking M_0 of $\mathcal{P}_1$ (s. Figure 1) is $M_0 = (0, \varepsilon, \varepsilon, \varepsilon)$.

It is clear that a time marking can change into an other one by firing a transition or by time elapsing as it is the case in each kind of time dependent PNs. First, we define the notion *ready to fire* and afterwards the notions *change by firing* and *change by time elapsing*.

Definition 2.5. (ready to fire)
Let M be a time marking with $M(p) = a_1^p a_2^p \ldots a_{|m(p)|}^p$ for each $p \in P$ and let t be an arbitrary transition in the tw-PN $\mathcal{P}$. Transition t is ready to fire in M, iff

(i) $t^- \leq m_M$,

(ii) $\forall p(p \in {}^\bullet t \longrightarrow \forall j(j \in \{1, \ldots, t^-(p)\} \longrightarrow l_p \leq a_j^p \leq u_p))$.

Definition 2.6. (firing a transition)
Let $\mathcal{P}$ be a tw-PN, let T be its set of transitions and let M be an arbitrary time marking in $\mathcal{P}$. The transition $t \in T$ can fire in the time marking M, iff t is ready to fire in M. After firing it, M changes into the time-marking M', denoted by $M \xrightarrow{t} M'$, which is defined as follows: Let $M(p) = a_1^p a_2^p \ldots a_n^p$ with $a_i^p \in \mathbb{R}_0^+$, $t^-(p) = k$ and $t^+(p) = r$[1]. Than it holds:

$$M'(p) := \begin{cases} a_{k+1}^p \ldots a_n^p 0^r & \text{if } k < n \\ 0^r & \text{if } k = n \end{cases} .$$

Remark 1. Let $M_1 \xrightarrow{t} M_2$ be a time marking change in the tw-PN $\mathcal{P}$. Obviously then, t is enabled in the marking m_{M_1} in the skeleton $S(\mathcal{P})$ and it holds: $m_{M_1} \xrightarrow{t} m_{M_2}$. □

[1]Let a be a letter in an alphabet. Then is $a^0 = \varepsilon$.

Definition 2.7. (time elapsing)
Let $\mathcal{P}$ be a tw-PN and let τ be a non-negative real number. Then, the elapsing of time τ in $\mathcal{P}$ is in any time marking always possible. Let M be an arbitrary time marking in $\mathcal{P}$. Then M is **changed** into the time marking M' **by the time elapsing** $\tau \in \mathbb{R}_0^+$, denoted by $M \xrightarrow{\tau} M'$, iff the following holds:

Let $M(p) = a_1^p a_2^p \dots a_n^p$ with $a_i^p \in \mathbb{R}_0^+$ and let $u_p < a_i + \tau$ and $a_{i+1} + \tau \leq u_p$ hold for each natural number i with $1 \leq i \leq n$. Then the succsessor time marking $M'(p) = b_1^p b_2^p \dots b_n^p$ is defined as the time marking in $\mathcal{P}$ with

$$b_j^p := \begin{cases} a_{i+j}^p + \tau & \text{if } i + j \leq n \\ (a_{i+j-n}^p + \tau)\ \widehat{mod}\ u_p & \text{else} \end{cases} .$$

Please note that:

$$a\ \widehat{mod}\ b := \begin{cases} a\ mod\ b & \text{if} \quad a\ mod\ b \neq 0 \\ b & \text{if} \quad a\ mod\ b = 0 \end{cases} .$$

Example 2. Let $\mathcal{P}$ be a tw-PN and let M be a time marking in $\mathcal{P}$ with

$$M(p) = 3.7\ \ 2.8\ \ 2.3\ \ 2\ \ 1.5\ \ 0.3\ \ 0.1 \text{ and } I(p) = (2, 6).$$

The succsessor time marking M' at the place p after the time elapcing $\tau = 4$ holds then:

$$M'(p) = 6\ \ 5.5\ \ 4.3\ \ 4.1\ \ 1.7\ \ 0.8\ \ 0.3$$

The behaviour of a given tw-PN $\mathcal{P} = (P, T, F, V, m_0, I)$ is defined by its changes from a given time marking into another. In general, the changes are an alternating series of time elapsings and firings. Thus, we use the following notions.

A *transition sequence* $\sigma = t_1 t_2 \dots t_n$ in $\mathcal{P}$ is a word in T^*. A *run* $\sigma(\tau)$ of a transition sequence $\sigma = t_1 t_2 \dots t_n$ with time elapcsings $\tau = \tau_0 \tau_1 \dots \tau_n \in (\mathbb{R}_0^+)^*$ is the word $\tau_0 t_1 \tau_1 \dots t_n \tau_n$ in $\mathbb{R}_0^+ (T\, \mathbb{R}_0^+)^*$. The *time-length* $l(\sigma(\tau))$ of the run $\sigma(\tau)$ is the sum $\tau_0 + \ldots + \tau_n$. A run $\sigma(\tau)$ is a *feasible* one in $\mathcal{P}$ if starting in M_0 all time marking changes defined by $\sigma(\tau)$ are possible in $\mathcal{P}$. A transition sequence σ is a *feasible transition sequence* in $\mathcal{P}$ if there exists at least a feasible run $\sigma(\tau)$.

A time marking M is called a *reachable time marking* in $\mathcal{P}$, if there exists a feasable run $\sigma(\tau)$ in $\mathcal{P}$ with $M_0 \xrightarrow{\sigma(\tau)} M$.

The *set of all reachable time markings* in $\mathcal{P}$, starting with time marking M, is denoted by $\mathcal{R}_{\mathcal{P}}(M)$. Please note that $\mathcal{R}_{\mathcal{P}}(M_0)$ is the set of all reachable time markings in $\mathcal{P}$. Finally, by $\mathcal{R}_{\mathcal{P}}(m_{M_0})$ we denote $\{m_M \mid M \in \mathcal{R}_{\mathcal{P}}(M_0)\}$.

A tw-PN $\mathcal{P}$ is *bounded* if the set $\{m_M \mid M \in \mathcal{R}_{\mathcal{P}}(M_0)\}$ is a finite one.

As already set, time elapsing is always possible in tw-PNs. Moreover, it can happen that in a time marking only time elapsing is possible. This can happen for two different reasons. First, there is no transition enabled in that time marking. Second, no enabled transition can become ready to fire because of the time restrictions. In the first case, the transition sequence, which leads to this time marking and ends here, is also a sequence in the skeleton. In the second case, we will call it *time-deadlock*, the underlying transition sequence is a firing sequence in the tw-PN, which can be continued (only) in the skeleton.

Definition 2.8. (time-deadlock)
Let M be a time marking in the tw-PN $\mathcal{P}$. The transition $\hat{t}$ is in M in a time-deadlock (short: t-DL), if

(i) $\hat{t}^{-} \leq m_M$

(ii) $\forall \tau (\tau \in \mathbb{R}_0^+ \longrightarrow M \xrightarrow{\tau} \overset{\hat{t}}{\not\rightarrow})$.

At the end of this section we introduce the notion *liveness* for tw-PNs. Actually, there are four levels of liveness. We consider here only the so called 4-liveness, defined by Lautenbach in [4]. This notion will also be defined in a similar manner to the definition for the classical PNs.

Definition 2.9. (liveness)
Let $\mathcal{P}$ be a tP-PN and M a reachable time marking.

(i) A transition t is live in the time marking M if
$\forall M'(M' \in \mathcal{R}_{\mathcal{P}}(M) \longrightarrow \exists M''(M'' \in \mathcal{R}_{\mathcal{P}}(M') \wedge M'' \xrightarrow{t})$

(ii) A tw-PN $\mathcal{P}$ is live if all transitions are live in M_0.

3. Reachability

In this section we discuss the reachability of an arbitrary tw-PN, i.e., we compare the reachabiltiy of tw-PNs with its skeleton. Our main goal is to show that both Petri nets have the same sets of reachable markings.

Finally, we briefly compare tw-PN and TPN. In particular, we make some remarks why one cannot consider natural numbers for the time elapsing only. Beyond this, we show that in a tw-PN, a feasible run can have "gaps" in its run.

Theorem 1. Let $\mathcal{P}$ be a tw-PN and $S(\mathcal{P})$ its skeleton. Then the firing sequence σ is a firing sequence in $S(\mathcal{P})$ if and only if σ is a firing sequence in $\mathcal{P}$.

Proof:
$(\Longleftarrow)$ This part of the proof is easy to see as a transition in $\mathcal{P}$ can fire if it is enabled. Therefore, it is enabled in $S(\mathcal{P})$.

$(\Longrightarrow)$ The idea of proof is to wait until the clock of each token reaches a time which is equal to the upper bound of the place where it is situated. After this occures, a transition is fired. This firing approach is called *ultimo rule*.

For this proof let $\sigma := t_1 \dots t_n$ a firing sequence in $S(\mathcal{P})$. Then we have $m_0 \xrightarrow{t_1} m_1 \xrightarrow{t_2} \dots \xrightarrow{t_n} m_n$ in $S(\mathcal{P})$.

We have to show that there exists a feasible run $\sigma(\tau) = \tau_0 t_1 \tau_1 \dots \tau_{n-1} t_n$ in $\mathcal{P}$. We make the proof by induction.

Basis: let $n = 1$, i.e. $\sigma = t_1$.

First, we need to know how much time must elapse until each token is "old" enough to become ready to fire. This is the case when the upper bound is not infinity, the time of the token is equal to the upper bound of the place, and when it is infinite, then to the lower bound.

Now for an arbitrary time marking M^* we define

$$U_{M^*} := \{u_p | p \in P \wedge u_p \neq \infty \wedge M^*(p) \neq \varepsilon\}$$
$$L_{M^*} := \{l_p | p \in P \wedge u_p = \infty \wedge M^*(p) \neq \varepsilon\}$$

and

$$B_{M^*} := U_{M^*} \cup L_{M^*}.$$

Let $\tau_0 :=$ LCM(B_{M_0}) and consider the time marking M_0' with $M_0 \xrightarrow{\tau_0} M_0'$. Then we have for each $p \in P$

$$M_0'(p) = \begin{cases} \varepsilon & \text{iff } m_0(p) = 0 \\ u_p^{m_0(p)} & \text{iff } m_0(p) \neq 0 \wedge u_p \neq \infty, \\ l_p^{m_0(p)} & \text{iff } m_0(p) \neq 0 \wedge u_p = \infty. \end{cases}$$

Obviously we have $m_0 = m_{M_0} = m_{M_0'}$ and therefore each transition t is enabled in $\mathcal{P}$ in M_0 iff t is enabled in $S(\mathcal{P})$ in $m_{M_0'}$. Furthermore, t is ready to fire in M_0' iff t is enabled in M_0'.

By assumption t_1 is enabled in $S(\mathcal{P})$ in $m_{M_0'}$ and thus ready to fire in $\mathcal{P}$ in M_0'. Let M_1 be the time marking we get by firing t_1, i.e., $M_0' \xrightarrow{t_1} M_1$. Then we have for each $p \in P$

$$M_1(p) = \begin{cases} \varepsilon & \text{iff } m_1(p) = 0 \\ u_p^{m_0(p)-|t_1^-(p)|} 0^{|t_1^+(p)|} & \text{iff } m_1(p) \neq 0 \wedge u_p \neq \infty, \\ l_p^{m_0(p)-|t_1^-(p)|} 0^{|t_1^+(p)|} & \text{iff } m_1(p) \neq 0 \wedge u_p = \infty. \end{cases}$$

Now we can see that the duration of dwell of each token in a place is equal either to the upper time of their place or 0 (if the upper is not infinity). Note that by definition 0 and u_p are in a way equivalent. Again we have $m_1 = m_{M_1}$.

Inductive step: Let $\sigma = t_1 \ldots t_{n-1} t_n$.

Let $\theta := t_1 \ldots t_{n-1}$. By the inductive hypothesis a feasible run $\theta(\tau) = \tau_0 t_1 \tau_1 \ldots \tau_{n-2} t_{n-1}$ exists with $M_o \xrightarrow{\theta(\tau)} M_{n-1}'$ and

$$M_{n-1}(p) = \begin{cases} \varepsilon & \text{iff } m_{n-1}(p) = 0, \\ u_p^{m_{n-2}(p)-|t_{n-1}^-(p)|} 0^{|t_{n-1}^+(p)|} & \text{iff } m_{n-1}(p) \neq 0 \wedge u_p \neq \infty, \\ w\, 0^{|t_{n-1}^+(p)|} & \text{iff } m_{n-1}(p) \neq 0 \wedge u_p = \infty, \end{cases}$$

where $w \in (\mathbb{R}_0^+)^*$ is a word such that for all letters (i.e. non-negative numbers) x which appear in w we have $x \geq l_p$.

Now consider the time $\tau_{n-1} :=$ LCM$(B_{M_{n-1}})$ and $M_{n-1} \xrightarrow{\tau_{n-1}} M_n'$ and it holds

$$M_{n-1}'(p) = \begin{cases} \varepsilon & \text{iff } m_{n-1}(p) = 0, \\ u_p^{m_{n-1}(p)} & \text{iff } m_{n-1}(p) \neq 0 \wedge u_p \neq \infty, \\ w & \text{iff } m_{n-1}(p) \neq 0 \wedge u_p = \infty. \end{cases}$$

Again a transition t is ready to fire in the certain time marking M_{n-1}' iff it is enabled in M_{n-1}'. By induction hypothesis the transition t_n is enabled in M_{n-1}' and therefore ready to fire in this time marking. That means, however, the transition sequence σ is a firing sequence in $\mathcal{P}$, as well. □

Corollary 1. For all tw-PN $\mathcal{P}$ the set $\mathcal{R}_{\mathcal{P}}(m_{M_0})$ is equal to the set $\mathcal{R}_{S(\mathcal{P})}$ of all reachable markings in the skeleton of $\mathcal{P}$.

When one compares the tw-PNs with the TPN (defined by Merlin) there are two more remarkable results.

First, we know that the TPN behave in the same manner as they do when describing time elapsing with real numbers or only natural numbers (cf. [10]). This is not true for tw-PNs. Let us consider the tw-PN $\mathcal{P}$ in Figure 2 and the transition sequence $\sigma = t_1 t_1$. After firing the run $0.5\ t_1\ 0.5\ t_1$ the only enabled transition t_2 is in t-DL and thus the run cannot be continued. In contrast, every run of σ where the time elapsings are natural numbers can be contiued.

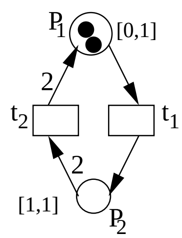


Figure 2. A tw-PN which "real" behaviour differs from its "natural" behaviour.

Second Let $\mathcal{Z}$ be a TPN and σ a transition sequence in $\mathcal{Z}$. Let $\sigma(\tau_\alpha)$ and $\sigma(\tau_\beta)$ be two feasable runs of a transition sequence σ with

$$l(\sigma(\tau_\alpha)) =: \alpha < \beta := l(\sigma(\tau_\beta)).$$

Then there exists a feasible run $\sigma(\tau_\gamma)$, with $l(\sigma(\tau_\gamma)) = \gamma$ for all $\gamma \in [\alpha, \beta]$.

This is not true for a tw-PN. It is possible that there are "time gaps" in the run. If there is a feasible run $\sigma(\tau_\alpha)$ and another feasible run $\sigma(\tau_\beta)$ so that $l(\sigma(\tau_\alpha)) = \alpha$ and $l(\sigma(\tau_\beta)) = \beta$ with $\alpha < \beta$, then there can be a real number $\gamma \in (\alpha, \beta)$ so that the length of all feasible runs $l(\sigma(\tau)) \neq \gamma$. Of course this is neither true for all α and β nor is it true for all runs.

To clarify this fact we state an example. Figure 3 shows the tw-PN we use in this example.

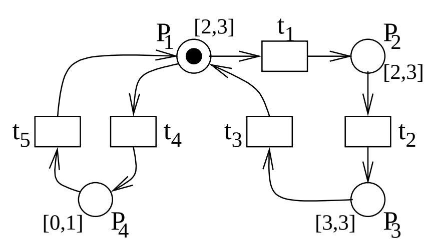


Figure 3. A tw-PN with a run with time gaps.

Now consider the sequence $\sigma = t_1\ t_2\ t_3$ and the feasible runs $\sigma(\tau_1) := 3\ t_1\ 3\ t_2\ 3\ t_3$ and $\sigma(\tau_2) := 5\ t_1\ 2\ t_2\ 3\ t_3$. It holds: $l(\sigma(\tau_1)) = 9$ and $l(\sigma(\tau_2)) = 10$. However, there is no feasible run $\sigma(\tau_3)$ such that $l(\sigma(\tau_3)) = 9.5$.

4. tw-PNs and Turing Machines

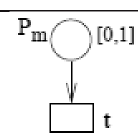
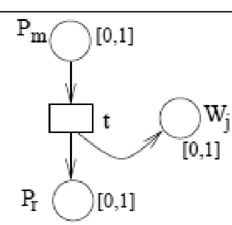
In this section we will show that tw-PN are not equivalent to Turing machines. In order to prove this we will show that this class of nets is not equivalent to counter machines. A counter machine itself is equivalent to Turing machines ([3]). Counter machines understand four different commands: START, HALT, INC (increase) and DEC (decrease). We will show how one can simulate the first three commands using tw-PN and, afterwards, we prove that one cannot simulate the DEC command with tw-PN. This proves that tW-PNs are not equivalent to counter machines and therefore they are not equivalent to Turing machines.

At first we will recall the four different commands of a counter machine.

Counter machine command	Description of the command
0 : START : m	Start the program and go to line m.
m : HALT	Stop the program.
m : INC(j) : r	Increase counter j by 1 then go to line r.
m : DEC(j) : r : s	If counter j equals 0 then go to line r else decrease counter j then go to line s.

The command DEC does two things. First, it checks the place for its emptiness (zero test) and, second, it subtracts a token from another place. We will see that the zero test is the problematic part.

Now we show how to simulate the first three commands (START, HALT and INC) with tw-PNs.

Notation of the numbered command	Model of the numbered command as a tw-PN
0 : START : m	
m : HALT	
m : INC(j) : r	

In order to simulate the DEC command it is nessecary to simulate the *zero test*. The zero test is a test that checks whether there is a token on a place p or not. If there is a token on p the place p_t gains a token, else the place p_f gains the token.

Theorem 2. The zero test cannot be simulated by a tw-PN.

Proof:
Assume that the zero test can be simulated for the place p by a tw-PN $\mathcal{P}_p$.

Then there exist two places p_t and p_f with $p_t \neq p_f$ so that $\mathcal{P}_p$ "stops" with either p_t or p_f marked, that means $\mathcal{P}_p$ reaches a marking where either p_t or p_f are marked and all transitions are disabled.

Case 1: $M_0(p) = \varepsilon$**.** Every feasible run $\sigma(\tau)$ in $\mathcal{P}_p$ with $M_0 \xrightarrow{\sigma(\tau)} M$ has only a finite amount of continuations so that for every continuation either case **(i)** or **(ii)** occurs

(i) $\exists t \in T$ and $t^- \leq m_M$

(ii) $\forall t \in T : t^- \not\leq m_M$ and $l(M(p_t)) = 0$ and $l(M(p_f)) = 1$.

Case 2: $M_0(p) \neq \varepsilon$**.** Every feasible run $\sigma(\tau)$ in $\mathcal{P}_p$ with $M_0 \xrightarrow{\sigma(\tau)} M$ has only a finite amount of continuations so that for every continuation either case **(i)** or **(ii)** occurs

(i) $\exists t \in T$ and $t^- \leq m_M$

(ii) $\forall t \in T : t^- \not\leq m_M$ and $l(M(p_t)) = 1$ and $l(M(p_f)) = 0$.

Please recall that $M_0(p) = \varepsilon$ really means that $l(M_0(p)) = 0$. Furthermore, note that cases **(ii)** can only happen once as no transition is any longer enabled.

Case 1: $M_0(p) = \varepsilon$

It follows that there is only a finite amount of feasible runs and they all stop in a time marking M with $l(M(p_f)) = 1$ and $l(M(p_t)) = 0$.

If there were another sequence in $S(\mathcal{P})$ we could reach it by using the ultimo property. Therefore, there is no further sequence σ in $S(\mathcal{P})$ so that σ is not feasible in $\mathcal{P}$.

Case 2: $M_0(p) \neq \varepsilon$

Analogous to Case 1.

Putting Case 1 and 2 together we receive that we can model the zero test in $S(\mathcal{P})$ and, therefore, the classical Petri nets are equivalent to Turing machines. That is a contradiction (cf. [9]). $\square$

5. Liveness

In this section we compare the liveness of tw-PNs with the liveness of its skeleton. As we have shown, the sets of the reachable markings of the both nets are equal, the liveness behaviour, however, of the both nets are not.

Some examples are shown here why the liveness behavior can differ between a tw-PN and its skeleton.

Lastly, we state a result for a class of restricted nets when the tw-PN is live. We will show furhermore the necessity for this restriction.

Remark 2. When a tw-PN $\mathcal{P}$ net is live then $S(\mathcal{P})$ is live as well. The opposite does not hold in general.

Proof:
This small example (figure 4) shows the tw-PN $\mathcal{P}_2$. It is obvious that $S(\mathcal{P}_2)$ is live. If we assume the sequence t_1 5.0 t_1, then it is easily seen that t_2 cannot become ready to fire. $\square$

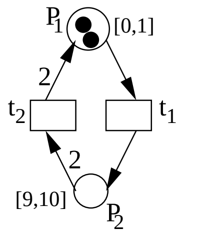


Figure 4. The tw-PN is not live although its skeleton is live.

Remark 3. Let $\mathcal{P}$ be a tw-PN with $V(f) = 1$ for each $f \in F$ and $|^{\bullet}t| \leq 1$. Then it is true: $\mathcal{P}$ is live iff $S(\mathcal{P})$ is live.

Remark 4. Note that restricting the net by $|^{\bullet}t| \leq 1$ is essential as Figure 5 shows.

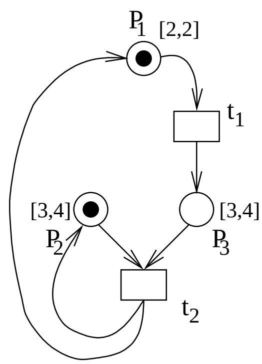


Figure 5. Example for a tw-PN $\mathcal{P}$ with $V(f) = 1\ \forall f \in F$ that is not live although $S(\mathcal{P})$ is live.

Remark 5. It is easy to see that a transition t with $^{\bullet}t = \{p\}$ is in time-DL in the time marking M if one of the following statements are true:

(i) For all time markings $M' = a'_1 \dots a'_n$ with $M \xrightarrow{\tau} M', \tau \in \mathbb{R}_0^+, V := V(p,t)$ the inequality $a'_n - a'_V > u_p - l_p$ holds.

(ii) The number of tokens in the interval $[l_p, u_p]$ is less than $V(p,t)$ for all time markings M' with $M \xrightarrow{\tau} M', \tau \in \mathbb{R}_0^+$.

Definition 5.1. (equidistant time marking)
Let $\mathcal{P}$ be a tw-PN. A time marking M is called an equidistant time marking in the place p if $M(p) = a_1^p, \dots, a_n^p$ with

$$a_{j+1}^p - a_j^p = \frac{u_p}{l(M(p))} \text{ for } j = 1, \dots, n-1 \quad \text{ and } (u_p - a_n^p) + a_1^p = \frac{u_p}{l(M(p))}.$$

The notion equidistant time marking is very important for the next proof. We will show that if a transition is not in a t-DL in an equidistance time marking, then it will be never in a t-DL in a time marking with the same number of tokens. This means that the existence of an equidistance time marking is the “worst case” for the occurrence of t-DL for a transition.

Theorem 3. Let $\mathcal{P}$ be a tw-PN and let t be a transition with ${}^{\bullet}t = \{p\}$. For each time markings M, M^* and M^e where $l(M^*(p)) = l(M^e(p)) = l(M(p))$ the following statements are equivalent:

(i) There is a time marking M^* so that t is in a time-DL in M^*.

(ii) t is in time-DL in the equidistant time marking M^e.

(iii) $l_p > u_p\left(1 - \frac{V(p,t)-1}{l(M(p))}\right)$.

Proof:
We set $n := l(M(p)) = l(M^*(p)) = l(M^e(p)), \lambda := V(p,t)$.

(ii) $\Longrightarrow$ (i) trivial

(iii) $\Longrightarrow$ (ii) We have

$$\begin{aligned} a_1 - a_\lambda &= (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{\lambda-1} - a_\lambda) \\ &= (\lambda - 1)\frac{u_p}{n} \\ &= u_p - u_p + (\lambda - 1)\frac{u_p}{n} \\ &= u_p - u_p\left(1 - \frac{\lambda - 1}{n}\right) > u_p - l_p \end{aligned}$$

and subsequently t is in M^e in a time-DL.

(i) $\Longrightarrow$ (iii) W.l.o.g. we can assume that $l_p, u_p \in \mathbb{N}$ and $a_n^p = u_p$ for the time marking M^*.

For an easier formal writing we copy the interval $[0, u_p]$ and add it to the end, i.e. we consider the time marking $\widetilde{M}^*(p) := \widetilde{a}_1^p, \ldots, \widetilde{a}_n^p, \widetilde{a}_1^p + u_p, \ldots, \widetilde{a}_n^p + u_p$.

Now let $\chi_{j,k}$ be the number of tokens in the interval $[j,k]$.

Furthermore there exists a natural number $\kappa := |\{a^p \in \widetilde{M}^*(p) : a \in \mathbb{N}\}|$. We assumed that $a_n^p = u_p$ thus $\kappa \geq 1$.

Therefore we have

$$\begin{aligned} n &= \sum_{j=0}^{u_p-1} \chi_{j,j+1} - \kappa & (1) \\ &= \frac{\sum_{j=0}^{u_p-1} \chi_{j,u_p-l_p+j} - \kappa}{u_p - l_p} \\ &\leq \frac{\sum_{j=0}^{u_p-1} (\lambda - 1) - \kappa}{u_p - l_p} \\ &< \frac{\sum_{j=0}^{u_p-1} (\lambda - 1)}{u_p - l_p} & (2) \\ &= \frac{u_p(\lambda - 1)}{u_p - l_p} \end{aligned}$$

where equality (1) holds as κ tokens are counted twice and inequality (2) holds as $\kappa \geq 1$.

And thus

$$\begin{aligned} & n < \frac{u_p}{u_p - l_p}(\lambda - 1) \\ \Longrightarrow \quad & u_p - l_p < \frac{u_p}{n}(\lambda - 1) \\ \Longrightarrow \quad & l_p > u_p - u_p\left(\frac{\lambda - 1}{n}\right) = u_p\left(1 - \frac{\lambda - 1}{n}\right). \end{aligned}$$

□

It is easy to conclude from Theorem 3, that if a pre-place of a transition contains more than a certain amount of tokens on a place, then the transition is able to fire. The exact number of tokens is also easy ascertainable. This is only true, however, for the class of restricted nets that we examined. The following shows another reason why we had to restrict the nets.

The Theorem 3 cannot be extended to transitions with $|^\bullet t| > 1$. The problem is that the number of tokens cannot be bound from above as it is done in the proof of case ($(i) \Rightarrow (iii)$). Figure 6 illustrates this fact.

The sequence $2t_1 2t_2 2t_1 2t_2 \ldots 2t_1 2t_2$ leads to arbitrary many tokens on each place but the transition t_3 cannot be made ready to fire.

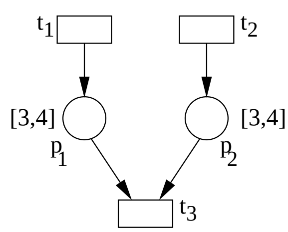


Figure 6. Illustration that pre-places of a transition t can hold arbitrary many tokens while t_3 is not ready to fire.

Lemma 5.1. Let $\mathcal{P}$ be a tw-PN and M a time marking in $\mathcal{P}$, t an arbitrary transition in $\mathcal{P}$ and t enabled in M. Furthermore, let $|^\bullet t| \leq 1$ and let the following estimate hold for each p with $(p,t) \in F$:

$$l_p \leq \frac{u_p}{V(p,t)}$$

Then t can become ready to fire in M.

Proof:
Let $\lambda := V(p,t)$. We know that t is enabled in M and thus we have $M(p) := n \geq \lambda$. Therefore

$$\begin{aligned} l_p &\leq \frac{u_p}{\lambda} \\ &= \frac{\lambda - \lambda + 1}{\lambda} u_p \\ &= \left(1 - \frac{\lambda - 1}{\lambda}\right) u_p \\ &\leq \left(1 - \frac{\lambda - 1}{n}\right) u_p \end{aligned}$$

and the rest follows from previous theorem. □

Let $P^* := \{p \in P \mid p^\bullet = \emptyset\}$ in the following.

Corollary 2. Let $\mathcal{P}$ be a tw-PN and let $|{}^\bullet t| \leq 1$ for every transition $t \in T$. Furthermore, let the following estimate hold for each $p \in P \setminus P^*$ and $t \in T$:

$$l_p \leq \frac{u_p}{\max\limits_{t \in p^\bullet} V(p,t)}.$$

Then there is no time marking M in $\mathcal{P}$ and no transition $t \in T$ so that t is in t-DL in M.

The reason why we need $\max\limits_{t \in p^\bullet} V(p,t)$ is simple. We must make sure that in any place p with $|p^\bullet| > 1$ we are able to fire any transition at any time.

Theorem 4. Let $\mathcal{P}$ be a tw-PN and let $S(\mathcal{P})$ be its skeleton so that $|{}^\bullet t| \leq 1$ for every $t \in T$ and so that $S(\mathcal{P})$ is live. Furthermore, let the following estimate hold for all places $p \in P \setminus P^*$:

$$l_p \leq \frac{u_p}{\max\limits_{t \in p^\bullet} V(p,t)}. \tag{3}$$

Then $\mathcal{P}$ is live.

Proof:
It follows immediately from Corollary 2 that no time marking is reachable in $\mathcal{P}$ so that a transition is in a t-DL in it. Eventually, it is evident that any transition sequence in $S(\mathcal{P})$ is also a transition sequence in $\mathcal{P}$ because of the Theorem 1. Therefore, the liveness of $\mathcal{P}$ can be derived from the liveness of $S(\mathcal{P})$. $\square$

Please note that Theorem 4 gives us a sufficient condition only. Figure 7 gives us an example for a tw-PN that violates the conditions in Theorem 4 but still is live.

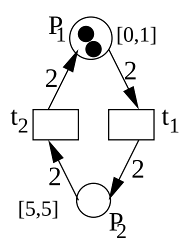


Figure 7. A live tw-PN that violates the estimate 3 of Theorem 4

Corollary 3. Let $\mathcal{P}$ be a tw-PN and let $S(\mathcal{P})$ be live. Let $|{}^\bullet t| \leq 1$ for every $t \in T$. Furthermore $l_p = 0$ or $u_p = \infty$ is true for each place p in $\mathcal{P}$. Then $\mathcal{P}$ is live, too.

6. Conclusion

In this paper we have presented a PN with time restrictions at the places. Usually time dependent PN are equivalent to the Turing machins. However, we have shown that the power of this class of time dependent PNs is equivalent to the power of the classical PNs and, therefore, not equivalent to the Turing machines. The set of all reachable markings of an arbitrary tw-PN is the same as the set of all reachable markings of its skeleton. The liveness behaviours of the both nets are different.

For a restricted class of nets we could show that the liveness behaviour is the same.

After all we surmise that the following property is true: Let $\mathcal{P}$ be an arbitrary tp-PN and let $S(\mathcal{P})$ be live and let the following estimate be true for all places $p \in P$:

$$l_p \leq \frac{u_p}{\sum\limits_{t \in p^\bullet} V(p,t)}.$$

Then $\mathcal{P}$ is live.

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References

[1] Parosh Aziz Abdulla, Pritha Mahata, and Richard Mayr. Dense-timed petri nets: Checking zenoness, token liveness and boundedness. *The Journal of Logical Methods in Computer Science*, 2007.

[2] Hans-Michael Hanisch. Analysis of place/transition nets with timed arcs and its application to batch process control. In *Application and Theory of Petri Nets*, pages 282–299, 1993.

[3] John E. Hopcroft, R. Motwani, and J. D. Ullman. *Einführung in die Automatentheorie, formale Sprachen und Komplexitätstheorie*. Pearson Studium, 2002.

[4] Kurt Lautenbach. *Exakte Bedingungen der Lebendigkeit für eine Klasse von Petri-Netzen*. Phd thesis, GMG, Bonn, 1973.

[5] Karsten Lemmer. *Diagnose diskreter modellierter Systeme mit Petrinetzen*. Dissertation, Technische Universität, Braunschweig, 1995.

[6] Morgan Magnin, Didier Lime, and Olivier (H.) Roux. Symbolic state space of stopwatch petri nets with discrete-time semantics. In *The 29th International Conference on Application and Theory of Petri Nets and other models of concurrency (ICATPN 2008)*, volume 5062 of *Lecture Notes in Computer Science*, Xi'an, China, June 2008. Springer.

[7] Merlin, Ph.M. *A Study of the Recoverability of Computing Systems*. PhD thesis, University of California, Computer Science Dept., Irvine, January 1974.

[8] L. Popova-Zeugmann. Petri Nets with Time Restrictions. *Systems Analysis, Modeling, Simulation(SAMS)*, pages 13–20, 1993.

[9] Popova-Zeugmann, L. *Time and Petri Nets (in German)*. Habilitation Thesis, Humboldt University at Berlin, Berlin, 2007.

[10] Popova-Zeugmann, L. and Schlatter, D. Analyzing Path in Time Petri Nets. *Fundamenta Informaticae (FI) 37, IOS Press, Amsterdam*, pages 311–327, 1999.

[11] C. Ramchandani. Analysis of asynchronous concurrent systems by Timed Petri Nets. *Project MAC-TR 120, MIT*, February 1974.

[12] J. Sifakis. Performance evaluation of systems using nets. In Brauer, W., editor, *Lecture Notes in Computer Science: Net Theory and Applications, Proc. of the Advanced Course on General Net Theory of Processes and Systems, Hamburg, 1979*, volume 84, pages 307–319, Berlin, Heidelberg, New York, 1980. Springer-Verlag.

[13] Juacun Wang. *Timed Petri Nets*. Kluwer Academic Publishers, Boston, Dordrecht, London, 1998.