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On Time Petri Nets

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Abstract: This paper shows how to study boundedness and liveness of a finite Time Petri net in a discrete way by using its reachability graph. The computation of the reachability graph defined here is a recursive-enumerable process. The decidability of reachability for an arbitrary state in a finite and bounded Time Petri net is shown here.

Time Petri nets, the subject of this paper, are based on the classical Petri nets defined by *C. A. Petri* [10]. A (classical) Petri net is an abstract, formal model of a system with asynchronous components. The major use of Petri nets is the modelling of systems of events in which it is possible for some events to occur concurrently.

As classical Petri nets have limitations, they have been extended in numerous ways. Two Petri net based models for representing concurrent systems with temporal constraints are known as Time Petri nets ([9]) and Timed Petri nets ([12]).

Time Petri nets are Petri nets in which two times, a, and b ($0 \leqq a \leqq b$, $a \neq \infty$), are associated with each transition. The times a and b, for transition t, are relative to the moment at which t was last enabled. Assuming that t was last enabled at time c, then t may fire only during the interval $[c + a, c + b]$ and must fire at the time $c + b$ at the latest, unless it is disabled before by the firing of another transition. Firing a transition takes no time. With Time Petri nets, *Merlin* studies recoverability problems in computer systems and the design of communication protocols ([9]).

Timed Petri nets are obtained from Petri nets by associating a firing time to each transition of the net. The firing rule is modified considering the firing time and that a transition must fire as soon as it is enabled to. Timed Petri nets are mainly used for performance evaluation ([12], [14], [4], [5] etc.)

In this paper we consider Time Petri nets where the time is described by rational numbers. In the first section we give a short explanation of what Petri nets are. In Section 2 Time Petri nets are defined and some basic properties are shown. Integer-firing is studied in Section 3. The notion "reachability graph" for a Time Petri net is defined in a discrete way. Sections 4 and 5 deal with boundedness and liveness. In the last section we study the reachability of an arbitrary state in a finite delay and bounded Time Petri net.

1. Petri nets. Notations

As usual, we will use the following notations:

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}^+$ is $\mathbb{N} \setminus \{0\}$. $\mathbb{Q}_0$ is the set of nonnegative rational numbers and $\mathbb{R}$ is the set of real numbers. T^* denotes the language of all words over the alphabet T, including the empty word e. Further, let f be a mapping where $f\colon A \to B \times C$.

By $\underline{f}(a)$ we denote the first component of $f(a)$ and by $\bar{f}(a)$ the second one.

Let g be a given function from A to B. Then, the symbol \$ defined a special value from the set $g(A)$. As usual, ω denotes the cardinal number of the set $\mathbb{N}$. $[\![a]\!]$ is the maximal integer, not greater than a, and $\ll a \gg$ is defined as $a - [\![a]\!]$. The other notations used in this paper are the same as in [15].

Definition 1.1. The structure $N = (P, T, F, V, m_0)$ is called a *(marked) Petri net* iff

(1) P, T, F are finite sets with
$P \cap T = \emptyset$,
$P \cup T \neq \emptyset$,
$F \subseteq (P \times T) \cup (T \times P)$,
$\mathrm{dom}(F) \cup \mathrm{cod}(F) = P \cup T$; (net)
(2) $V: F \to \mathbb{N}^+$; (weight of the arcs)
(3) $m_0: P \to \mathbb{N}$. (initial marking)

The elements $p \in P$ are called *places* and $t \in T$ *transitions*. For each place $p \in P$ the set $Fp = \{t | t \in T \wedge tFp\}$ respectively $pF = \{t | t \in T \wedge pFt\}$ is the set of the *pretransitions* respectively *posttransitions* of p. Analogously, the set Ft respectively tF is called the set of the *preplaces* resp. *postplaces* or *inputplaces* resp. *outputplaces* of t.

Definition 1.2. An injective mapping $m: P \to \mathbb{N}$ is called a *marking* of N.

Definition 1.3. Each transition $t \in T$ *induces* the markings t^- and t^+, which are defined as follows:

$$t^-(p) := \begin{cases} V(p, t) & \text{iff } (p, t) \in F \\ 0 & \text{otherwise} \end{cases}$$

and

$$t^+(p) := \begin{cases} V(t, p) & \text{iff } (t, p) \in F \\ 0 & \text{otherwise}. \end{cases}$$

By Δt we denote $t^+ - t^-$.

Definition 1.4. Let $N = (P, T, F, V, m_0)$ be a Petri net, $t \in T$, and m be a marking. The transition t *is enabled (may fire) by* m iff $t^- \leq m$ (cf. [15, p. 55]).

Definition 1.5. Let $N = (P, T, F, V, m_0)$ be a Petri net and let t be an enabled transition by the marking m. m' is called the *follower marking* of m iff $m' = m + \Delta t$.

Here we have defined the Petri net and some closely related notions. A detailed characterization of these nets is explained e.g. in [13], [15], [6].

2. Time Petri net, state

In Petri nets, the time is involved only in the sense of a temporal sequence. Problems of priorities and coercion to fire are not solvable here, in general. However, such constraints can be well modelled by Time Petri nets. These problems are very often contained in real systems.

Definition 2.1. The structure ($Z = (P, T, f, V, m_0, I)$ is called a *(marked) Time Petri net* iff:
(1) (P, T, F, V, m_0) is a (marked) Petri net.
(2) $I: T \to \mathbb{Q}_0 \times \mathbb{Q}_0 \cup \{\infty\})$ with $\forall t(t \in T \to (\underline{I}(t) \leqq \bar{I}(t) \wedge \underline{I}(t) < \infty))$.

By Z^N we denote the Petri-net (P, T, F, V, m_0). I is called the *time function* of Z, $\underline{I}(t)$ and $\bar{I}(t)$ the *earlest firing time (EFT)* and *latest firing time (LFT)* of the transition t, respectively.

In the following we will consider nets with a finite set of transitions.

Obviously, it is easier to study the behaviour of such time Petri nets whose EFT and LFT are natural numbers. It may be shown that, for each TPN, we can find another TPN whose EFT and LFT are natural numbers. We call such two nets time equivalent.

Definition 2.2. Let $Z_i = (P_i, T_i, F_i, V_i, m_{0_i}, I_i)$ with $i = 1, 2$ be two TPN. Z_1 and Z_2 are *time equivalent* (notation: $Z_1 \underset{c}{\Rightarrow} Z_2$) iff:

(1) $Z_1^N = Z_2^N$.
(2) There exists such a constant $c \neq 0$ that for each $t \in T$:
(2.1) $\bar{I}_1(t) = \infty \leftrightarrow \bar{I}_2(t) = \infty$,
(2.2) $\underline{I}_1(t) = 0 \leftrightarrow \underline{I}_2(t) = 0$ and $\bar{I}_1(t) = 0 \leftrightarrow \bar{I}_2(t) = 0$,
(2.3) $\dfrac{\underline{I}_1(t)}{\underline{I}_2(t)} = c$ iff $\underline{I}_2(t) \neq 0$,
(2.4) $\dfrac{\bar{I}_1(t)}{\bar{I}_2(t)} = c$ iff $\bar{I}_2(t) \neq 0$.

Definition 2.2 implies that the time unit in Z_2 is c-times shorter than the time unit in Z_1.

Theorem 2.1. *Let Z_1 be a TPN. Then there exists a TPN Z_2 with $I_2: T \to \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ and Z_1 and Z_2 are time equivalent.*

Ideas of the proof. At first we compute the L.C.M. r of the denominators of all EFT_1 and LFT_1 of the given TPN Z_1. Then, we define EFT_2 resp. LFT_2 as the products of the EFT_1 resp. LFT_1 with r.

In the following we always consider time functions I with $\text{cod}(I) \subseteq (\mathbb{N} \times \mathbb{N} \cup \{\infty\})$.

Definition 2.3. Let $Z = (P, T, F, V, m_0, I)$ be a TPN. The mapping $m: P \to \mathbb{N}$ is called a *marking* in Z.

It is very easy to see that the total behaviour of a Time Petri net cannot be described by the markings only. For this reason we define the notion of a "state". By a state we will understand the markings in dependence on the time. Before, we introduce the notion of a "conflict".

A more detailed discussion about conflicts can be found in [15], [13], [11].

Definition 2.4. Let Z be a TPN. Two transitions t_1 and t_2 *are in conflict* iff $Ft_1 \cap Ft_2 \neq \emptyset$.

Definition 2.5. Let $Z = (P, T, F, V, m_0, I)$ be a TPN and $J: T \to \mathbb{Q}_0 \cup \{\$\}$. $\mathfrak{z} = (m, J)$ is called a *state* in Z iff:
(1) $m \in R_N(m_0)$ (m is a reachable marking in Z^N),

(2) $\forall t(t \in T \wedge t^- \leqq m \rightarrow J(t) \leqq \bar{I}(t))$ and
(3) $\forall t(t \in T \wedge t^- \nleqq m \rightarrow J(t) = \$)$.
We denote the function J as the *situation of the watch.*

How can we interpret a state in a Time Petri net now? In the net each transition t has a watch. The watch does not work ($J(t) = \$$) at the marking m if t is disabled at m.
If t is enabled at m, the watch of t shows the time ($J(t)$), elapsed since t was enabled. Of course, this time is less that or equal to $\bar{I}(t)$.

Definition 2.6. Let $Z = (P, T, F, V, m_0, I)$ be a TPN and let $\mathfrak{z}_0 = (m_0, J_0)$ be a state in Z. $\mathfrak{z}_0$ is called an *initial state* of Z iff

$$J_0(t) := \begin{cases} 0 & \text{iff} \quad t^- \leqq m_0 \\ \$ & \text{otherwise} \end{cases}$$

for each transition t.

Definition 2.7. Let $Z = (P, T, F, V, m_0, I)$ be a TPN. The transition t is *ready to fire* in the state $\mathfrak{z} = (m, J)$ iff:
(1) $t^- \leqq m$ (t is enabled by m in Z^N),
(2) $\underline{I}(t) \leqq J(t)$.

The definition above means that a transition t in a TPN Z is ready to fire at the marking m if t is enabled in the PN Z^N at the marking m and the watch of t shows the EFT of t or a later time.
Now, we will explain the firing rules for the TPN.

Definition 2.8. Let $Z = (P, T, F, V, m_0, I)$ be a TPN. The state (m, J) changes into the state (m', J') by the time duration $\tau \in \mathbb{Q}_0$ (notation: $(m, J) \overset{e}{\underset{\tau}{\rightarrow}} (m', J')$) iff:
(1) $m = m'$,
(2) $\forall t(t \in T \wedge J(t) \neq \$ \rightarrow J(t) + \tau \leqq \bar{I}(t))$,

$$(3)\ J'(t) = \begin{cases} J(t) + \tau & \text{iff} \quad t^- \leqq m \\ \$ & \text{otherwise}. \end{cases}$$

Definition 2.9. The state (m, J) *changes into the state* (m', J') *by firing of* $\hat{t}$ (notation: $(m, J) \overset{\hat{t}}{\rightarrow} (m', J')$) iff:
(1) $\hat{t}$ is ready to fire in (m, J),
(2) $m' = m + \Delta\hat{t}$,

$$(3)\ J'(t) = \begin{cases} \$ & \text{iff} \quad t^- \nleqq m \\ J(t) & \text{iff} \quad t^- \leqq m \wedge t^- \leqq m' \wedge Ft \cap F\hat{t} = \emptyset \\ 0 & \text{otherwise}. \end{cases}$$

The Time Petri net notion is more powerful than the PN notion. In PN we cannot test whether an unbounded place is marked, but we can do so in TPN (cf. Fig. 1). This gives us the possibility to simulate counter machines with TPN (cf. [15, p. 102–107]).

Definition 2.10. Let Z be a time Petri net. The smallest set of states $\mathfrak{Z}_Z(\mathfrak{z}_0)$ which involves the state $\mathfrak{z}_0$ and is closed under the operations "change of states" in the sense of Definition 2.8 and 2.9 is called the *state structure* of Z.

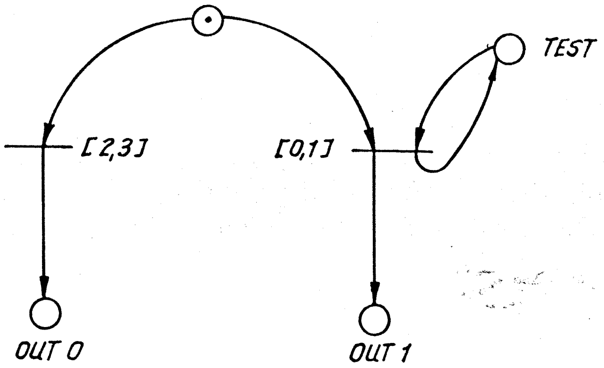

Fig. 1

Clearly, the state structure of a Time Petri net is the set of all states which are reachable from the initial state $\mathfrak{z}_0$.

Definition 2.11. Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net. $R_Z(m_0)$ is called the *set of all reachable markings* in Z iff

$$R_Z(m_0) := \{m \mid \exists J\ ((m, J) \in \mathfrak{Z}_Z(\mathfrak{z}_0))\}\,.$$

Obviously, the inclusion $R_Z(m_0) \subseteqq R_{Z^N}(m_0)$ is true.

Definition 2.12.

(A). Let $\mathfrak{z}, \mathfrak{z}'$, be two states in the Time Petri net $Z = (P, T, F, V, m_0, I)$, $w \in T^*$ a sequence of transitions, and $\xi \in \mathbb{Q}_0^{\mathrm{len}\,(w)+1}$. Then the state $\mathfrak{z}$ *changes into the state* $\mathfrak{z}'$ *by* w *and* ξ (notation: $\mathfrak{z} \xrightarrow[\xi]{w} \mathfrak{z}'$) iff

Basis: $w = e$, $\quad \mathfrak{z} : \mathfrak{z} \xrightarrow[\xi]{e} \mathfrak{z}$.

Step: $\mathfrak{z} \xrightarrow[\xi\tau]{wt} \mathfrak{z}'$ iff $J\mathfrak{z}'' J\mathfrak{z}''' (\mathfrak{z} \xrightarrow[\xi]{w} \mathfrak{z}'' \wedge \mathfrak{z}'' \xrightarrow{t} \mathfrak{z}''' \wedge \mathfrak{z}''' \xrightarrow[\tau]{e} \mathfrak{z}')$.

(B). Let $Z = (P, T, F, V, m_0, I)$ be a TPN and $\mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0)$. Then the sequence $w \in T^*$ *can fire at* $\mathfrak{z}$ in Z (notation: $\mathfrak{z} \xrightarrow[*]{w}$) if there exists a sequence of rational numbers ξ and a state $\mathfrak{z}' \in \mathfrak{Z}_Z(\mathfrak{z}_0)$ and $\mathfrak{z} \xrightarrow[\xi]{w} \mathfrak{z}'$.

(C). Let $Z = (P, T, F, V, m_0, I)$ and $\mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0)$. Then the sequence of transitions $w \in T^*$ *can integer-fire at* $\mathfrak{z}$ (notation: $\mathfrak{z} \xrightarrow[*-\mathrm{i}]{w}$) iff there exist a sequence ξ of integers such that $\mathfrak{z} \xrightarrow[\xi]{w} \mathfrak{z}'$.

An important fact of the Time Petri nets is the following one: The Church-Rosser property is valid in some, but not in all classes of Time Petri nets. In the net shown in Fig. 2 the Church-Rosser property does not hold.

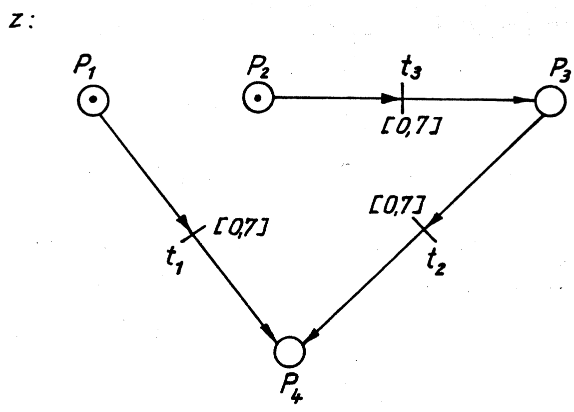

Fig. 2

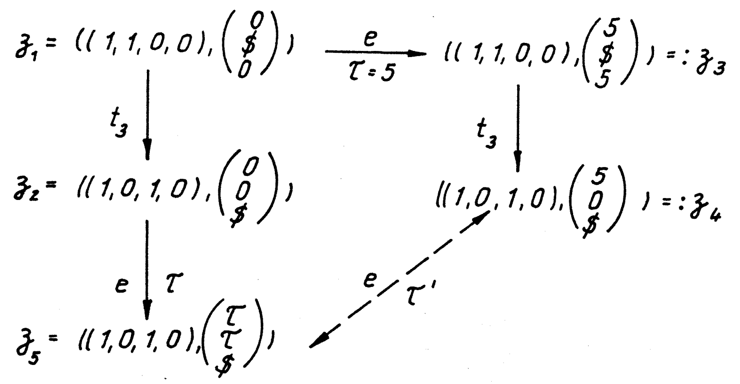

Fig. 3

There does not exist a $\tau' \in \mathbb{Q}_0$ with

$$\mathfrak{z}_4 \xrightarrow[\tau']{e} \mathfrak{z}_5 \quad \text{or} \quad \mathfrak{z}_5 \xrightarrow[\tau']{e} \mathfrak{z}_4 .$$

Definition 2.13 and Theorem 2.2 consider a class of Time Petri nets where t Church-Rosser property is valid:

Definition 2.13. Let $\mathfrak{M}$ be the set of all Time Petri nets with the property: If tw transitions are enabled in a state then they are in conflict and have the same EFT (forma $\mathfrak{M} = \{Z \mid \forall \mathfrak{z}\, \forall t_1\, \forall t_2 (\mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0) \wedge t_1 \in T \wedge t_2 \in T \wedge t_1, t_2$ are enabled at $\mathfrak{z} \rightarrow Ft_1 \cap F$ $\neq \emptyset \wedge \underline{I}(t_1) = \underline{I}(t_2))\}$).

Theorem 2.2. *Let* $Z = (P, T, F, V, m_0, I)$ *be a Time Petri net from the class* $\mathfrak{M}$ *and let*
a) *a state* (m, J), $w \in T^*$, $\tau' \in \mathbb{Q}_0^{\mathrm{len}(w)}$ *with*

$$(m, J) \xrightarrow[\tau']{w} (m', J'')$$

b) $\forall \hat{t}(\hat{t} \in T \wedge J(\hat{t}) \neq \$ \rightarrow J(\hat{t}) < \bar{I}(\hat{t}))$
be given.
Then:

$$\forall \tau(\tau \in \mathbb{Q}_0 \wedge \tau > 0 \wedge (m, J) \xrightarrow[\tau]{e} (m, J') \xrightarrow[\tau'']{w} (m', J''') \rightarrow$$
$$\exists \tau''' \exists J'''((m, J') \xrightarrow[\tau'']{w} (m', J''') \wedge (m', J'') \xrightarrow[\tau''']{e} (m', J''')) .$$

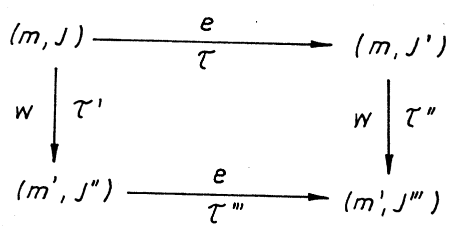

Fig. 4

We can prove the proposition by induction on len(w) (cf. [11]).

3. Integer-fire

In this section we will show that each reachable state is integer-reachable, too.

Definition 3.1. Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net. Z is said to be of *finit* *delay* iff

$$\forall t(t \in T \rightarrow \bar{I}(t) < \infty) .$$

In the sequel we will consider Time Petri nets of finite delay (FDTPN).

Definition 3.2. Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net and let $\mathfrak{z} = (m, J)$ be a state from $\mathfrak{Z}_Z(\mathfrak{z}_0)$. $\mathfrak{z}$ is called an integer-state iff:

$$\forall t(t \in T \wedge t^- \leqq m \rightarrow J(t) \in \mathbb{N}).$$

Time Petri nets have an infinite number of states in general. However, only a finite number of integer-states belongs to each marking of an FDTPN.

Our aim is to show that each enabled transition t is integer-enabled. More precisely: If $\mathfrak{z}_0 \rightarrow \mathfrak{z}_1 \rightarrow \dots \rightarrow \mathfrak{z}_n \overset{t}{\rightarrow}$ is given then there exist integer-states $\hat{\mathfrak{z}}_i$, $i = 1, \dots, n$, with

$$\mathfrak{z}_0 = \hat{\mathfrak{z}}_0 \rightarrow \hat{\mathfrak{z}}_1 \rightarrow \dots \rightarrow \hat{\mathfrak{z}}_n \overset{t}{\rightarrow}.$$

By the way, we will see that the integer-state $\hat{\mathfrak{z}}_i$ is situated "near" the state $\mathfrak{z}_i$.

Lemma 3.1. *Let a and b be two nonnegative numbers with $a \geqq b$. Then, it holds:*

$$[\![a - b]\!] \leqq [\![a]\!] - [\![b]\!] \leqq [\![a - b]\!] + 1.$$

Proof. Let $a = [\![a]\!] + \langle\!\langle a \rangle\!\rangle$ and $b = [\![b]\!] + \langle\!\langle b \rangle\!\rangle$. Then

$$[\![a - b]\!] = [\![\underbrace{[\![a]\!] - [\![b]\!]}_{\in \mathbb{N}} + \langle\!\langle a \rangle\!\rangle - \langle\!\langle b \rangle\!\rangle]\!] = (1)$$

Case 1. $\langle\!\langle a \rangle\!\rangle > \langle\!\langle b \rangle\!\rangle$. — Then:

$$0 \leqq \langle\!\langle a \rangle\!\rangle - \langle\!\langle b \rangle\!\rangle < 1$$

and consequently

$$(1) = [\![a]\!] - [\![b]\!],$$

i.e.

$$[\![a - b]\!] = [\![a]\!] - [\![b]\!].$$

Hence

$$[\![a - b]\!] = [\![a]\!] - [\![b]\!] < [\![a - b]\!] + 1$$

is true.

Case 2. $\langle\!\langle a \rangle\!\rangle < \langle\!\langle b \rangle\!\rangle$. — Then

$$-1 < \langle\!\langle a \rangle\!\rangle - \langle\!\langle b \rangle\!\rangle < 0$$

and consequently it holds

$$(1) = [\![a]\!] - [\![b]\!] - 1,$$

i.e.

$$[\![a - b]\!] = [\![a]\!] - [\![b]\!] - 1 < [\![a]\!] - [\![b]\!].$$

Hence, the inequality

$$[\![a - b]\!] < [\![a]\!] - [\![b]\!] = [\![a - b]\!] + 1$$

holds in this case.

Thus, the lemma is true.

Lemma 3.2. *Let τ_i be a nonnegative rational number for each i. Further, let*

$$f_0 := \tau_0\,, \qquad \tilde{\tau}_0 := [\![\tau_0]\!]\,,$$

$$f_i := \sum_{j=0}^{i} \tau_j\,, \qquad i = 0, \ldots, n\,, \qquad f_{-1} := 0\,,$$

$$\tilde{\tau}_i := \left[\!\!\left[f_i - \sum_{j=0}^{i=1} \tilde{\tau}_j \right]\!\!\right] \quad \text{and} \quad \tilde{f}_j := \sum_{j=0}^{i} \tilde{\tau}_j\,.$$

$i \geq 1$

f_i – the actual global time
$\tilde{f}_i$ – the actual global integer time
($\tilde{f}_i \leq f_i$)

Then, we have

(1) $\tilde{\tau}_i = [\![f_i]\!] - \tilde{f}_{i-1}$ *for each* $i = 0, \ldots, n$;

(2) $\tilde{f}_i = [\![f_i]\!]$ *for each* $i = 0, \ldots, n$;

(3) $\sum_{j=k}^{l} \tilde{\tau}_j = \tilde{f}_l - \tilde{f}_{k-1} = [\![f_l]\!] - [\![f_{k-l}]\!]$ *for* $0 \leqq k \leqq l$;

(4) *if* $\left(\sum_{j=k}^{l} \tilde{\tau}_j\right) \in \mathbb{N}$ *then* $\sum_{j=k}^{l} \tilde{\tau}_j = \sum_{j=k}^{l} \tau_j$;

(5) $\left[\!\!\left[\sum_{j=k}^{l} \tau_j\right]\!\!\right] \leqq \sum_{j=k}^{l} \tilde{\tau}_j \leqq \left[\!\!\left[\sum_{j=k}^{l} \tau_j\right]\!\!\right] + 1\,.$

Proof.

(1) It is true:

$$\tilde{\tau}_i := \left[\!\!\left[f_i - \sum_{j=0}^{i-1} \tilde{\tau}_j \right]\!\!\right] = [\![f_i - \underbrace{\tilde{f}_{i-1}}_{\in \mathbb{N}}]\!] = [\![f_i]\!] - \tilde{f}_{i-1}\,.$$

(2) For each $i = 1, \ldots, n$ it holds:

$$\tilde{f}_i = \sum_{j=0}^{i-1} \tilde{\tau}_j + \tilde{\tau}_i = \tilde{f}_{i-1} + \tilde{\tau}_i \underset{(1)}{=} \tilde{f}_{i-1} + [\![f_i]\!] - \tilde{f}_{i-1} = [\![f_i]\!]\,.$$

(3) It is true:

$$\sum_{j=k}^{l} \tilde{\tau}_j = \sum_{j=0}^{l} \tilde{\tau}_j - \sum_{j=1}^{k-1} \tilde{\tau}_j \underset{\substack{\text{acc.}\\ \text{def.}}}{=} \tilde{f}_l - \tilde{f}_{k-1} \underset{(2)}{=} [\![f_l]\!] - [\![f_{k-1}]\!]\,.$$

(4) We have:

$$\sum_{j=k}^{l} \tilde{\tau}_j \underset{(3)}{=} [\![f_l]\!] - [\![f_{k-l}]\!] = \left[\!\!\left[\sum_{j=0}^{k-1} \tau_j + \underbrace{\sum_{j=k}^{l} \tau_j}_{\substack{\in \mathbb{N}\\ \text{according}\\ \text{to the}\\ \text{supposition}}} \right]\!\!\right] - \left[\!\!\left[\sum_{j=1}^{k-1} \tau_j \right]\!\!\right]$$

$$= \left[\!\!\left[\sum_{j=0}^{k-1} \tau_j \right]\!\!\right] + \sum_{j=k}^{l} \tau_j - \left[\!\!\left[\sum_{j=o}^{k-1} \tau_j \right]\!\!\right] = \sum_{j=k}^{l} \tau_j\,.$$

Because of the def. of f_i it holds:

$[\![\sum_{j=k}^{\ell} \tau_\ell]\!] = [\![\sum_{j=0}^{\ell} \tau_j - \sum_{j=0}^{k-1} \tau_j]\!] = [\![f_\ell - f_k]\!]$

(5) According to Lemma 3.1, it holds:

$$[\![f_\ell - f_{k-1}]\!] \leqq [\![f_\ell]\!] - [\![f_{k-1}]\!] \leqq [\![f_\ell - f_{k-1}]\!] + 1 .$$

Furthermore, we have:

$$[\![f_l]\!] - [\![f_{k-1}]\!] \underset{\substack{\text{acc.}\\ \text{to (2)}}}{=} \tilde{f}_l - \tilde{f}_{k-1} \underset{\substack{\text{acc.}\\ \text{to (3)}}}{=} \sum_{j=k}^{l} \tilde{\tau}_j .$$

Hence, (5) is true.

Theorem 3.1. *Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net and let*

$$\mathfrak{z}_0 \xrightarrow[\tau_0]{} \mathfrak{z}_0^* \xrightarrow{t_0} \mathfrak{z}_1 \xrightarrow[\tau_1]{} \mathfrak{z}_1^* \xrightarrow{t_1} \mathfrak{z}_2 \cdots \mathfrak{z}_n \xrightarrow[\tau_n]{} \mathfrak{z}_n^* \xrightarrow{t_n} .$$

Then, the following is true:

For each $i = 1, \ldots, n$ there exist integer-states $\mathfrak{z}_i := (m_i', J_i')$, $\mathfrak{z}_i'' := (m_i'', J_i'')$ with

(1) $\mathfrak{z}_0' \xrightarrow[\tilde{\tau}_0]{} \mathfrak{z}_0'' \xrightarrow{t_0} \mathfrak{z}_1' \xrightarrow[\tilde{\tau}_1]{} \mathfrak{z}_1'' \xrightarrow{t_1} \mathfrak{z}_2' \cdots \mathfrak{z}_n' \xrightarrow[\tilde{\tau}_n]{} \mathfrak{z}_n'' \xrightarrow{t_n}$

(2) $m_i' = m_i''$, $\quad m_i'' = m_i^* = m_i$

(3) *t is ready to fire in $\mathfrak{z}_i''$ if t is ready to fire in $\mathfrak{z}_i^*$*

(4) $t^- \leqq m_i^* \rightarrow [\![J_i^*(t)]\!] \leqq J''(t) \leqq [\![J_i^*(t)]\!] + 1$

Proof. The theorem will be proved by induction on n. Let $\mathfrak{z}_0' := \mathfrak{z}_0$.

Basis. $i = 0$. $\tilde{\tau}_i$ are constructed as in Lemma 3.2.

ad (1): We have

$$\mathfrak{z}_0' \xrightarrow[\tilde{\tau}_0]{} \mathfrak{z}_0'' \xrightarrow{t_1} .$$

Obviously, $\mathfrak{z}_0'$ and $\mathfrak{z}_0''$ are integer-states.

Further, it holds:

$$J_0''(t) = \tilde{\tau}_1 = [\![\tilde{\tau}_1]\!] \leqq \tau_1 \leqq \bar{I}(t) ,$$

i.e.

$$\forall t (t^- \leqq m_0 \rightarrow J_0''(t) \leqq \bar{I}(t)) .$$

ad (2): Because of $\mathfrak{z}_0' = \mathfrak{z}_0$

$$m_0'' = m_0' = m_0 = m_0^*$$

is true.

ad (3): Let t be ready to fire in $\mathfrak{z}^*$, that means

$$J_0^*(t) \geqq \underline{I}(t) \quad \text{and} \quad \underline{I}(t) \leqq J_0^*(t) = \tau_0 .$$

Due to $\underline{I}(t) \in \mathbb{N}$ and the inequality above, $\underline{I}(t) \leqq [\![\tau_0]\!] = J_0''(t)$ follows, that means t is ready to fire in $\mathfrak{z}_0''$.

ad (4): Let $t^- \leqq m_0^* = m_0$. Obviously, here it is true that

$$[\![J_0^*(t)]\!] = [\![\tau_0]\!] = J_0''(t) \leqq [\![\tau_0]\!] + 1 = [\![J_0^*(t)]\!] + 1 .$$

Step. We assume that (1), (2), (3) and (4) are true for i and show that they are true for $i + 1$, too.

ad (1): The existence of $\mathfrak{z}''_{i+1}$ has to be shown where

$$\mathfrak{z}'_{i+1} \xrightarrow[\tilde{\tau}_{i+1}]{} \mathfrak{z}''_{i+1} \, .$$

Let $t \in T$ be a transition with $t^- \leqq m_{i+1}$. Therefore:

$$J_{i+1}(t) + \tau_{i+1} = J^*_{i+1}(t) \leqq \bar{I}(t) \, .$$

Case 1. $J^*_{i+1}(t) < \bar{I}(t)$.

Then $[\![J^*_{i+1}(t)]\!] + 1 \leq \bar{I}(t)$ ⊃

$$[\![J_{i+1}(t) + \tau_{i+1}]\!] + 1 \leqq \bar{I}(t) \, , \tag{1}$$

because $\bar{I}(t) \in \mathbb{N}$.

Let $J_{i+1}(t) = \sum_{j=k}^{i+1} \tau_j$, that means t became enabled at m_{k-1} and since then it has kept this property. Hence, by the induction hypothesis it holds that

$$J''_i(t) = \sum_{j=k}^{i+1} \tilde{\tau}_j \, .$$

Hence, it follows

$$J''_{i+1}(t) = J'_{i+1}(t) + \tilde{\tau}_{i+1} = \sum_{j=k}^{i+1} \tilde{\tau}_j \underset{\text{Lemma 3.2}}{\leqq} [\![\sum_{j=k}^{i+1} \tau_j]\!] + 1 = [\![J_{i+1}(t) + \tau_{i+1}]\!] + 1 =$$
$$= 1 + [\![J^*_{i+1}(t)]\!] \leqq \bar{I}(t) \, .$$

Case 2. $J^*_{i+1}(t) = \bar{I}(t)$.

Then $J^*_{i+1}(t)$ is a natural number and therefore it holds that

$$\sum_{j=k}^{i+1} \tau_j \in \mathbb{N} \, .$$

That means:

$$J''_{i+1}(t) = J'_{i+1}(t) + \tilde{\tau}_{i+1} = \sum_{j=k}^{i+1} \tilde{\tau}_j \underset{\text{Lemma 3.2}}{=} \sum_{j=2}^{i+1} \tau_j$$
$$= J^*_{i+1}(t) = \bar{I}(t) \, .$$

ad (2): Obvious.

ad (3): Let t^* be ready to fire in $\mathfrak{z}^*_{i+1}$. Then: $t^- \leqq m_{i+1}$ and $\underline{I}(t) \leqq J^*_{i+1}(t) = \sum_{j=k}^{i+1} \tau_j$, i.e. $\underline{I}(t) \leqq \sum_{j=k}^{i+1} \tau_j$. Since $\underline{I}(t) \in \mathbb{N}$, it holds: $\underline{I}(t) \leqq \left[\!\!\left[\sum_{j=k}^{i+1} \tau_j \right]\!\!\right]$.

Now, by Lemma 3.25, it follows that

$$\left[\!\!\left[\sum_{j=k}^{i+1} \tau_j \right]\!\!\right] \leqq \sum_{j=k}^{i+1} \tilde{\tau}_j = \tilde{J}_{i+1}(t) \, ,$$

i.e. t is ready to fire in $\mathfrak{z}''_{i+1}$, too.

ad (4): The result follows by Lemma 3.2.5.

Theorem 3.1 is very important for the theory of Time Petri nets of finite delay. Then, due to this theorem it is possible to make a discrete analysis of boundedness and liveness.

4. Boundedness

Definition 4.1. The graph $EG_Z(\mathfrak{z}_0)$ is called a *reachability graph* of the Time Petri net Z iff its nodes are the integer-states from $\mathfrak{Z}_Z(\mathfrak{z}_0)$ and its arcs are defined by the triples $(\mathfrak{z}, \tau, \mathfrak{z}')$ resp. $(\mathfrak{z}, t, \mathfrak{z}')$, where $\mathfrak{z} \xrightarrow{\tau} \mathfrak{z}'$ resp. $\mathfrak{z} \xrightarrow{t} \mathfrak{z}'$.

Definition 4.2. Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net.

a) A place $p \in P$ is called *bounded* (at $\mathfrak{z}_0$) iff there exists a natural number K with $m(p) \leqq K$ for each marking $m \in R_Z(\mathfrak{z}_0)$,

b) the net Z is *bounded* (at $\mathfrak{z}_0$) iff all places p are bounded (at $\mathfrak{z}_0$).

According to [15] the reachability graph of a Petri net is finite if and only if the net is bounded. This result is true for Time Petri nets of finite delay, too (see the example given in Figs. 5 and 6).

Theorem 4.1. *Let* $Z = (P, T, F, V, m_0, I)$ *be an FDTPN. Z is bounded if and only if* $EG_Z(\mathfrak{z}_0)$ *is finite.*

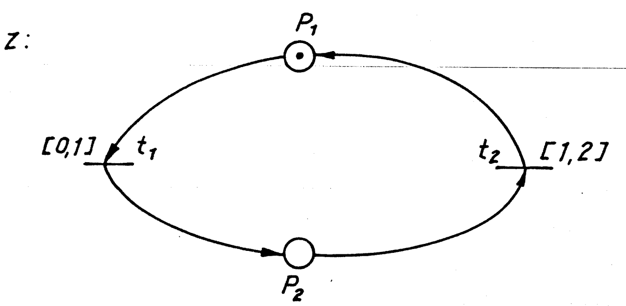

Fig. 5

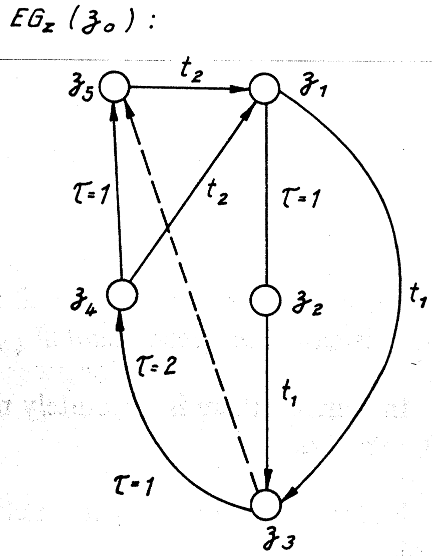

Fig. 6

$$\mathfrak{z}_1 = \left((1,0), \begin{pmatrix} 0 \\ \$ \end{pmatrix}\right),$$

$$\mathfrak{z}_2 = \left((1,0), \begin{pmatrix} 1 \\ \$ \end{pmatrix}\right),$$

$$\mathfrak{z}_3 = \left((0,1). \begin{pmatrix} \$ \\ 0 \end{pmatrix}\right),$$

$$\mathfrak{z}_4 = \left((0,1), \begin{pmatrix} \$ \\ 1 \end{pmatrix}\right),$$

$$\mathfrak{z}_5 = \left((0,1), \begin{pmatrix} \$ \\ 2 \end{pmatrix}\right)$$

Proof. (necessity). Let Z be a bounded net. Then the set $R_Z(m_0)$ is finite. Hence, the set

$$\{\mathfrak{z} | \mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0) \wedge \mathfrak{z} \text{ an integer-state}\}$$

is finite, too. Consequently, the set of the nodes of the reachability graph $EG_Z(\mathfrak{z})$ is finite, too.

(sufficiency). Let Z be a Time Petri net of finite delay and let $EG_Z(\mathfrak{z}_0)$ be finite. Then, the set of markings reachable from $\mathfrak{z}_0$ by integer-states only, is finite. According to Theorem 3.1, $R_Z(\mathfrak{z}_0)$ is finite, too. This implies (Definition 4.1) that Z is bounded.

Thus, we proved that the boundedness of FDTPN is recursive-enumerable. It is not difficult to prove that boundedness is invariant under time-equivalence.

5. Liveness in Time Petri nets

Definition 5.1. Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net, $\mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0)$ and $t \in T$.

1. t is called *live in* $\mathfrak{z}$ iff:

$$\forall \mathfrak{z}'(\mathfrak{z}' \in \mathfrak{Z}_Z(\mathfrak{z}) \to \exists \mathfrak{z}''(\mathfrak{z}'' \in \mathfrak{Z}_z(\mathfrak{z}') \wedge t \text{ is ready to fire in } \mathfrak{z}'').$$

2. t is called *dead in* $\mathfrak{z}$ iff:

$$\forall \mathfrak{z}'(\mathfrak{z}' \in \mathfrak{Z}_Z(\mathfrak{z}) \to t \text{ is not ready to fire in } \mathfrak{z}').$$

3. $\mathfrak{z}$ is called *live in* Z iff all transition $t \in T$ are live in $\mathfrak{z}$.
4. $\mathfrak{z}$ is called *dead in* Z iff each transition $t \in T$ is dead in $\mathfrak{z}$.
5. t is called *live* resp. *dead* in Z iff t is live resp. dead in $\mathfrak{z}_0$.
6. Z is called *live* resp. *dead* iff $\mathfrak{z}_0$ is live resp. dead in Z.

In general, there is absolutely no relation between liveness in a Time Petri net Z and in the Petri net Z^N.

Examples. For the Time Petri nets given in Figs. 7, 8 and 9, the following assertions hold true:

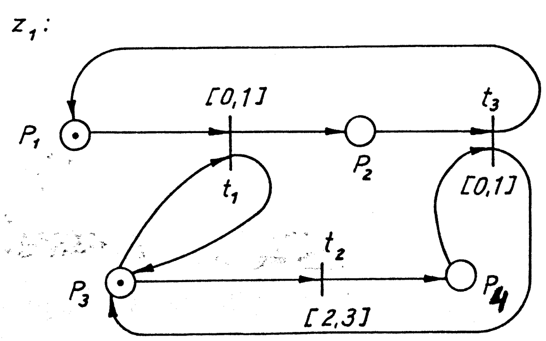

Fig. 7. Time Petri net Z_1

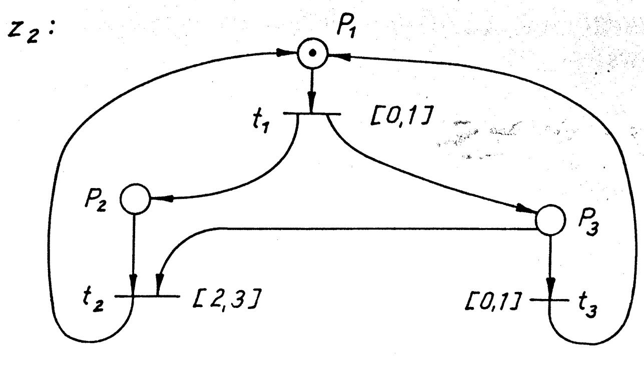

Fig. 8. Time Petri net Z_2

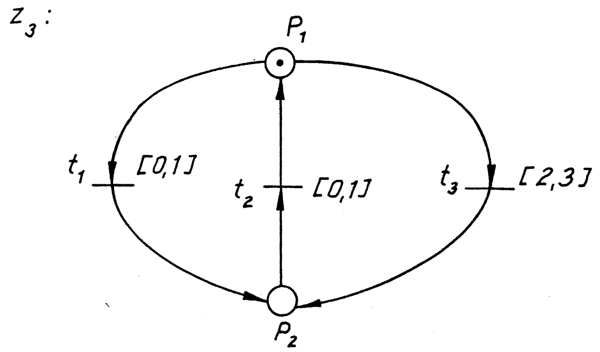

Fig. 9. Time Petri net Z_3

1) t_2 is live in $\mathfrak{z}_0 = (m_0, J_0)$ in Z_1;
   t_2 is not live at m_0 in Z_1^N.
2) t_2 is not live in $\mathfrak{z}_0 = (m, J_0)$ in Z_2;
   t_2 is live at m_0 in Z_2^N.
3) t_3 is dead in $\mathfrak{z}_0$ in Z_3; t_3 is not dead at m_0 in Z_3^N.
4) t_2 is live in Z_1;
   t_2 is not live in Z_1^N.
5) t_2 is not live in Z_2;
   t_2 is live in Z_2^N.
6) $\mathfrak{z}_0$ is live in Z_1;
   m_0 is not live in Z_1^N.
7) $\mathfrak{z}_0$ is not live in Z_2;
   m_0 is live in Z_2^N.
8) Z_1 is live;
   Z_1^N is not live.
9) Z_2 is not live;
   Z_2^N is live.

Analogously to Petri nets, the proof of the liveness of Time Petri nets can be carried out on the reachability graph of the net. For this reason we define the notion of a "language of a Time Petri net".

Definition 5.2. $L_{Z(\mathfrak{z}_0)}$ is called the *language of the Time Petri net* $Z = (P, T, F, V, m_0, I)$ iff

$$L_{Z(\mathfrak{z}_0)} := \{w \mid w \in T^* \wedge \mathfrak{z}_0 \xrightarrow[*]{w}\}.$$

Definition 5.3. The *language* $L_{EG_Z(\mathfrak{z}_0)}$ *of the reachability graph* $EG_Z(\mathfrak{z}_0)$ of the Time Petri net Z is defined as follows:

$$L_{EG_Z(\mathfrak{z}_0)} := \{w | w \in T^* \land \mathfrak{z}_0 \xrightarrow[*-i]{w}\}.$$

Definition 5.4. The language $L_{Z(\mathfrak{z}_0)}$ resp. $L_{EG_Z(\mathfrak{z}_0)}$ is called *live* iff

$$\forall w \forall t (w \in L_Z \text{ resp. } L_{EG_Z(\mathfrak{z}_0)} \land t \in T$$
$$\rightarrow \exists u (u \in T^* \land wut \in (L_Z \text{ resp. } L_{EG_Z(\mathfrak{z}_0)}))).$$

Theorem 5.1. *Let* $Z = (P, T, F, V, m_0, I)$ *be a Time Petri net. Then*

$$L_{Z(\mathfrak{z}_0)} = L_{EG_Z(\mathfrak{z}_0)}.$$

Proof. ($\rightarrow$) $L_{Z(\mathfrak{z}_0)} \subseteq L_{EG_Z(\mathfrak{z}_0)}$ follows immediately by Theorem 3.1.
($\leftarrow$) The inequality

$$L_{EG_Z(\mathfrak{z}_0)} \subseteq L_{Z(\mathfrak{z}_0)}$$

follows by Definitions 5.2 and 5.3.

Theorem 5.2. *Let Z be a Time Petri net.* L_Z *is live if and only if Z is live.*

Proof. (necessity). Let the language L_Z be live. We will show that the Time Petri net Z is live, too.

Let $t \in T$ be a transition and let $\mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0)$ be a state. It is sufficient to show that there exists a state $\mathfrak{z}' \in \mathfrak{Z}_Z(\mathfrak{z}_0)$ and that t is ready to fire in $\mathfrak{z}'$.

Because of $\mathfrak{z} \in \mathfrak{Z}_Z(\mathfrak{z}_0)$, there exists a $w \in T^*$ with $\mathfrak{z}_0 \xrightarrow{w} \mathfrak{z}$. According to Definition 5.2 it holds: $w \in L_{Z(\mathfrak{z}_0)}$.

Since $L_{Z(\mathfrak{z}_0)}$ is live, there exists a $u \in T^*$ such that $wut \in L_{Z(\mathfrak{z}_0)}$. Consequently, there exist states $(\mathfrak{z}_i, \ldots, \mathfrak{z}_{i_{len(u)}})$ with

$$\mathfrak{z} \xrightarrow[\tau_1]{t_{i_1}} \mathfrak{z}_{i_1} \xrightarrow[\tau_2]{t_{i_2}} \cdots \xrightarrow[\tau_r]{t_{i_r}} \mathfrak{z}_{i_r} \xrightarrow[\tau]{e} \mathfrak{z}' \xrightarrow{t}.$$

Hence, $\mathfrak{z}' \in \mathfrak{Z}_{Z(\mathfrak{z})}$ and t is ready to fire in $\mathfrak{z}'$.

(sufficiency). Let Z be a live Time Petri net. Now, we want the language $L_{Z(\mathfrak{z}_0)}$ to become live, too. Let $w \in L_{Z(\mathfrak{z}_0)}$ and $t \in T$ be given. Then, it is sufficient to show that there is a $u \in T$ with $w \in L_{Z(\mathfrak{z}_0)}$. Since $w \in L_{Z(\mathfrak{z}_0)}$ and according to Definition 5.2 there exists a state

$$\mathfrak{z} \in \mathfrak{Z}_{Z(\mathfrak{z}_0)} \quad \text{with} \quad \mathfrak{z}_0 \xrightarrow[*]{w} \mathfrak{z}.$$

Furthermore, Z is live. As $\mathfrak{z} \in \mathfrak{Z}_{Z(\mathfrak{z}_0)}$ is valid and according to Definition 5.1, there exists a state $\mathfrak{z}' \in \mathfrak{Z}_{Z(\mathfrak{z})}$ with t ready to fire in $\mathfrak{z}'$.

Let $t_{i_1}, \ldots, t_{i_n}$ be such that

$$\mathfrak{z} \xrightarrow[*]{t_{i_1} \cdots t_{i_n}} \mathfrak{z}' \xrightarrow{t}.$$

We define: $u := t_{i_1} \ldots t_{i_n}$. Therefore, it holds: $u \in T^*$ and

$$\mathfrak{z} \xrightarrow[*]{u} \mathfrak{z}' \xrightarrow{t}. \qquad (2)$$

By (1) and (2), it follows that $\mathfrak{z}_0 \xrightarrow[*]{w} \mathfrak{z} \xrightarrow[*]{u} \mathfrak{z} \xrightarrow{t}$ and, hence, $L_{Z(\mathfrak{z}_0)}$ is live (Definition 5.2 and 5.4).

Theorem 5.3. *Let $Z = (P, T, F, V, m_0, I)$ be a Time Petri net. Then it holds: Z is live if and only if*

$\forall t(t \in T \rightarrow$
(1) $\exists(\mathfrak{z}, t, \mathfrak{z}')$ $((\mathfrak{z}, t, \mathfrak{z}')$ *is an arc in* $\mathrm{EG}_Z(\mathfrak{z}_0)))$ $\wedge$
(2) $\forall\mathfrak{z}(\mathfrak{z}$ *is a node in* $\mathrm{EG}_Z(\mathfrak{z}_0) \rightarrow$ [redacted]$\exists\sigma($[redacted] $\wedge\ \sigma = (\mathfrak{z}\sigma^*\mathfrak{z}^*\mathfrak{z}^{**})$ *path in* $\mathrm{EG}_Z(\mathfrak{z}_0)$
and $(\mathfrak{z}^*, t, \mathfrak{z}^{**})$ *arc in* $\mathrm{EG}_Z(\mathfrak{z}_0))$.

The proof follows immediately by Definitions 5.3 and 5.4.

Examples demonstrating the properties shown in this paper are to be found in [11].

6. Reachability

The subject of this section is the reachability of an arbitrary state in a Time Petri net. If the given state is not an integer one, we will choose a new unit of measuring the time, which is by so many times smaller as the time unit before so, that with the new time unit the state is an integer state. Then, using the reachability graph of the net we can decide whether the state is reachable or not.

Let $Z = (P, T, F, V, m_0, I)$ be an FDTPN and $\mathfrak{z} = (m, J)$ a state. We will transform Z into the (time equivalent) TPN Z^* as follows: Let

$$J(t) = \begin{cases} \$ & \text{iff } t^- \nleqq m \\ \dfrac{p_t}{q_t} & \text{iff } t^- \leqq m, \end{cases}$$

where $p_t, q_t \in \mathbb{N}_0$.

Let $r := \text{L.C.M.}\left\{\dfrac{p_t}{q_t} \middle| t^- \leqq m\right\}$.

Now we consider I^* such that

$$\underline{I}^*(t) := \underline{I}(t) \cdot r,$$

$$\bar{I}^*(t) := \bar{I}(t) \cdot r.$$

Obviously, it holds: $I^*(t) \leqq \mathbb{N} \times \mathbb{N}$ for each $t \in T$. Let Z^* be the TPN (P, T, F, V, m_0, I^*).

Further, let

$$J^*(t) := \begin{cases} \$ & \text{iff } J(t) = \$ \\ J(t) \cdot r & \text{otherwise} \end{cases}$$

hold for each $t \in T$.

It is clear that

$$J^*: T \rightarrow \mathbb{N} \cup \{\$\}.$$

Theorem 6.1. *Let Z an FDTPN, $\mathfrak{z}$ a state and Z^* and $\mathfrak{z}^*$ be defined like above. Then it holds: $\mathfrak{z}$ is reachable in Z if and only if $\mathfrak{z}^*$ is reachable in Z^*.*

Proof. ($\rightarrow$) Let $\mathfrak{z}$ be reachable in Z. We aim at showing that $\mathfrak{z}^*$ is reachable in Z^*.

Since $\mathfrak{z}$ is reachable in Z, there exist states $\mathfrak{z}_0, \mathfrak{z}_1, \hat{\mathfrak{z}}_1, \ldots, \mathfrak{z}_n, \hat{\mathfrak{z}}_n$, transitions $t_1, \ldots, t_n$ and time-durations $\tau_0, \ldots, \tau_n \in \mathbb{Q}_0$, and it holds:

$$\mathfrak{z}_0 \xrightarrow[\tau_0]{} \mathfrak{z}_1 \xrightarrow{t_1} \hat{\mathfrak{z}}_1 \xrightarrow[\tau_1]{} \mathfrak{z}_2 \xrightarrow{t_2} \hat{\mathfrak{z}}_2 \xrightarrow[\tau_2]{} \cdots \xrightarrow[\tau_{n-1}]{} \mathfrak{z}_n \xrightarrow{t_n} \hat{\mathfrak{z}}_n \xrightarrow[\tau_n]{} \mathfrak{z} \tag{1}$$

in Z.

Now, we will consider the states $\mathfrak{z}_0^*, \mathfrak{z}_i^*, \hat{\mathfrak{z}}_i^*$ and the time-durations τ_i^*, which are defined as follows:

$$\mathfrak{z}_i^* := (m_{\mathfrak{z}_i}, J_i^*), \qquad \hat{\mathfrak{z}}_i^* := (m_{\hat{\mathfrak{z}}_i}, \hat{J}_i^*), \qquad \tau_i^* := \tau_i \cdot r,$$

where

$$J_i^*(t) := \begin{cases} \$ & \text{iff} \quad J_i(t) = \$ \\ J_i(t) \cdot r & \text{otherwise}, \end{cases}$$

$$\hat{J}_i^*(t) := \begin{cases} \$ & \text{iff} \quad J_i(t) = \$ \\ J_i(t) \cdot r & \text{otherwise}. \end{cases}$$

Obviously, $J_i^*(t)$ and $\hat{J}_i^*(t)$ have the value \$ or they are rational numbers, and the time durations τ_i^* are rational numbers, too. Furthermore, it is clear that $\mathfrak{z}_0^* = \mathfrak{z}_0$. Now, it is not difficult to see, that the sequence

$$\mathfrak{z}_0^* \xrightarrow[\tau_0^*]{} \mathfrak{z}_1^* \xrightarrow{t_1} \hat{\mathfrak{z}}_1^* \xrightarrow[\tau_1^*]{} \mathfrak{z}_2^* \xrightarrow{t_2} \hat{\mathfrak{z}}_2^* \xrightarrow[\tau_2^*]{} \cdots \xrightarrow[\tau_{n-1}^*]{} \mathfrak{z}_n^* \xrightarrow{t_n} \hat{\mathfrak{z}}_n^* \xrightarrow[\tau_n^*]{} \mathfrak{z}^* \tag{2}$$

is a sequence in Z^*. Therefore, $\mathfrak{z}^*$ is reachable in Z^*.

($\leftarrow$) analogously.

In Section 4 we have shown that a bounded FDTPN has a finite reachability graph. Furthermore, for such nets we can prove whether an integer-state is reachable or not, that means, we can decide whether $\mathfrak{z}^*$ is reachable in Z^* or not and, therefore, according to the theorem above, we can decide whether $\mathfrak{z}$ is reachable in Z or not.

Further, the following property is true for a Time Petri net:

Let $\mathfrak{z}$ be an arbitrary state in a Time Petri net Z and let $\underline{\mathfrak{z}}$, $\bar{\mathfrak{z}}$ be such that

$$m_{\underline{\mathfrak{z}}} = m_{\bar{\mathfrak{z}}} = m_{\mathfrak{z}},$$

$$\underline{J}(t) = \begin{cases} \$ & \text{iff} \quad J(t) = \$ \\ [\![J(t)]\!] & \text{otherwise} \end{cases}$$

and

$$\bar{J}(t) := \begin{cases} \$ & \text{iff} \quad J(t) = \$ \\ [\![J(t)]\!] + 1 & \text{otherwise}. \end{cases}$$

It is obvious that $\underline{\mathfrak{z}}$ and $\bar{\mathfrak{z}}$ are integer-states. In [11] it is shown that if the state $\underline{\mathfrak{z}}$ or $\bar{\mathfrak{z}}$ is not reachable in Z, then $\mathfrak{z}$ is surely not reachable in Z, too. Therefore, by studying the reachability of a state $\mathfrak{z}$ in a finite and bounded net we suggest first to prove if $\underline{\mathfrak{z}}$ and $\bar{\mathfrak{z}}$ are reachable in Z. If this is true, then Z should be transformed into Z^* and $\mathfrak{z}$ into $\mathfrak{z}^*$. Now, the reachability of $\mathfrak{z}^*$ in Z^* can be decided and, according to the theorem, the reachability of $\mathfrak{z}$ in Z is decidable.

Concluding remarks

The result that the boundedness and liveness of a Time Petri net of finite delay can be studied by its reachability graph can be implemented in a very convenient way.

Acknowledgement

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Kurzfassung

In diesem Artikel ist ein diskreter Weg zur Berechnung der Beschränktheit und der Lebendigkeit eines finiten Zeit-Petri-Netzes (nach *Merlin*) mittels des Erreichbarkeitsgraphen des Netzes gezeigt worden. Die Berechnung des hier definierten Erreichbarkeitsgraphen is rekursiv-aufzählbar. Die Entscheidbarkeit der Erreichbarkeit eines beliebigen Zustandes in einem finiten und beschränkten Zeit-Petri-Netz wird bewiesen.

Резюме

В статье указан способ вычисления ограничености и живучести конечных временных сетей Петри (по *Мерлину*) при помощи их графов достижимости. Вычисление определенных в статье графов достижимости рекурсивно перечислимо. Доказана разрешимость достижимости произвольного состояния в конечных и ограниченных временных сетях Петри.

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