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Time-independent Liveness in Time Petri Nets

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Streszczenie. In this paper we consider a class of Time Petri nets defined by structural restrictions. Each Time Petri net which belongs to this class has the property that their liveness behaviour does not depend on the time. Therefore, the Time Petri net is live when its skeleton is live.

1. Introduction

Petri nets have been used to describe and study concurrent systems for more than forty-five years. At first glance, time and concurrence do not seem to have much in common. But if one looks closer, the opposite is the case. There are endless examples from different areas showing this. For this reason, a large variety of time dependent Petri nets have been introduced and well studied. One of the first such nets is the Time Petri net (TPN), introduced in [10].

TPNs are derived from classical Petri nets. Additionally, each transition t is associated with a time interval $[a_t, b_t]$. Here, a_t and b_t are relative to the time, when t was enabled last. When t becomes enabled, it cannot fire before a_t time units have elapsed, and it has to fire not later than b_t time units unless t was disabled in between by the firing of another transition. The firing of a transition itself takes no time. The time interval is designed by real numbers, but the interval bounds are nonnegative rational numbers. It is easy to see (cf. [4]) that w.l.o.g. the interval bounds can be considered as integers only. Thus, the interval bounds a_t and b_t of any transition t are natural numbers, including zero and $a_t \leq b_t$ or

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$b_t = \infty$. a_t is called *earliest firing time* of the transition t (short: $eft(t)$) and b_t, the *latest firing time* of t (short: $lft(t)$) .

Every possible situation in a given TPN can be described completely by a state $z = (m, h)$, consisting of a (place-) marking m and a transition marking h. The (place-) marking, which is a place vector (i.e. the vector has as many components as places in the considered TPN), is defined as the marking notion in classical Petri nets. The time marking, which is a transition vector (i.e. the vector has as many components as transitions in the considered TPN), describes the time circumstances in the considered situation. In general, each TPN has an infinite number of states. Thus, the central problem for analysis of a certain TPN is knowledge about its state space.

In [7] it is shown that the state space can be characterized parametrically and that knowledge about the reachable integer-states, i.e. states whose time markings are (nonnegative) integers, is sufficient to determine the entire behavior of the net at any point in time. In the case that some lfts = ∞, then a subset of all reachable integer-states, the so-called set of the essential-states, expresses the net behaviour (cf. [5]). A reachability graph $RG(Z)$ for a TPN Z can be defined in such a way that its vertices are the reachable integer-states or the reachable essential-states, respectively. The edges are defined by the triples (z, t, z') and (z, τ, z'), $\tau \in \mathbb{N}$, where $z \xrightarrow{t} z'$ and $z \xrightarrow{\tau} z'$, respectively. This graph is finite if and only if the set of the reachable markings of the net is finite. The calculation of a single integer-state is very easy.

Actually, a reachability graph for TPN was first introduced by Berthomieu and Menasche in [2] respectivaly Berthomieu and Diaz in [1]. They provide a method for analyzing the qualitative behavior of the net based on the computing of certain subsets of reachable states, called state classes. However, the essential-states method is exponentially better in worst case, but in the case that in a TPN the concurrence is rather low, then the state-classes method compute a smaller reachability graph.

A further way to analyze a TPN is the translation into a timed automaton and then to apply the analyzing algorithms used there (cf. [9]).

The most important behavioral properties of a TPN (and of a PN as well) are the reachability, the boundedness, the liveness, and the reversibility (cf. [11]). These properties are decidable for an arbitrary classical PN, but not for an arbitrary TPN in general. The reason for this is the nonequivalence of the classical PNs respectively the equivalence of the TPNs to the Turingmachines (cf.[6]). However, there are restricted classes of TPN for which the properties are decidable. In [6] three structural and one dynamical restricted classes of TPNs are given for those the liveness problem is equivalent to the liveness problem of its skeleton, which means the TPN considered without time, and therefore it is decidable. The first class is the set of all arbitrary TPNs with $lft(t) = 0$ for all transitions t. The next one is the set of all arbitrary TPNs with $lft(t) = \infty$ for all transitions t. The third structural restricted class is the set of TPNs which satisfied three conditions: 1. the skeleton is a generalized EFC net[1], 2. $lft(t) > 0$ for all transitions t, and 3. $\mathcal{M}in(p) < \mathcal{M}ax(p)$[1] for all places p.

This paper is organised as follows. In the next section we recall some basic notions. In the third section we introduce first some new notions and give several remarks. Afterwards, we proof some structural properties of the new defined class of PNs. This is followed by the proof of the main property that each TPN is time-independent live for which the skeleton is a generalized ES[1], $lft(t) > 0$ for all its transitions t, and $\mathcal{M}in(p) \leq \mathcal{M}ax(p)$[1] for all its places p. Finally, we summarize the results and give some remarks including future outlook.

[1]This notion is defined in section 3.1

2. Basic notations and definitions

We use the following notations in this paper: $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. $\mathbb{Q}_0^+$, resectively $\mathbb{R}_0^+$, is the set of nonnegative rational numbers, respectively the set of nonnegative real numbers . T^* denotes the language of all words over the alphabet T, including the empty word ε; $l(w)$ is the length of the word w. The "floor" of a real number r denoted by $\lfloor r \rfloor$ is the maximum of the set of integers that are not greater than r, respectively, the "ceiling" of r denoted by $\lceil r \rceil$ is minimum of the set of integers that are not smaller than r.

The definition of a (classical) Petri net is as follows:

Definition 2.1. The structure $\mathcal{N} = (P, T, F, V, m_o)$ is called a **Petri net (PN)** iff

1. P, T, F are finite sets with
   $P \cap T = \emptyset$, $P \cup T \neq \emptyset$, $F \subseteq (P \times T) \cup (T \times P)$ and $dom(F) \cup cod(F) = P \cup T$
2. $V : F \longrightarrow \mathbb{N}^+$ (weight of the arcs)
3. $m_o : P \longrightarrow \mathbb{N}$ (initial marking)

A **marking** of a PN is a function $m : P \longrightarrow \mathbb{N}$, such that $m(p)$ denotes the number of tokens at the place p. The **pre-sets** and **post-sets** of a transition t are given by ${}^\bullet t := \{p \mid p \in P \wedge (p, t) \in F\}$ and $t^\bullet := \{p \mid p \in P \wedge (t, p) \in F\}$, respectively. Analogously, the **pre-sets** and **post-sets** of a place p are given by ${}^\bullet p := \{t \mid t \in T \wedge (t, p) \in F\}$ and $p^\bullet := \{tp \mid t \in T \wedge (p, t) \in F\}$, respectively. Each transition $t \in T$ induces the marking t^- and t^+, defined as follows:

$$t^-(p) = \begin{cases} V(p,t) & \text{iff } (p,t) \in F \\ 0 & \text{iff } (p,t) \notin F \end{cases} \qquad t^+(p) = \begin{cases} V(t,p) & \text{iff } (t,p) \in F \\ 0 & \text{iff } (t,p) \notin F \end{cases}.$$

Moreover, Δt denotes $t^+ - t^-$. A transition $t \in T$ is **enabled (may fire)** at a marking m iff $t^- \leq m$ (i.e. $t^-(p) \leq m(p)$ for every place $p \in P$). When an enabled transition t at a marking m fires, this yields a new marking m' given by $m'(p) := m(p) + \Delta t(p)$ and denoted by $m \xrightarrow{t} m'$. Thus, the dynamical behavior of a classical PN is characterized by firing transitions that leads to change of the markings.

A marking m is a **reachable** one in $\mathcal{N}$ if there is a transition sequence which can fire starting at m_0 and ending at m. The set of all markings reachable in $\mathcal{N}$ is denoted by $R_\mathcal{N}$.

Definition 2.2. The structure $\mathcal{Z} = (P, T, F, V, m_o, I)$ is called a **Time Petri net (TPN)** iff

1. $S(\mathcal{Z}) := (P, T, F, V, m_o)$ is a PN.
2. $I : T \longrightarrow \mathbb{Q}_0^+ \times (\mathbb{Q}_0^+ \cup \{\infty\})$ and $I_1(t) \leq I_2(t)$ for each $t \in T$, where $I(t) = (I_1(t), I_2(t))$.

A TPN is called finite Time Petri net (FTPN) iff $I : T \longrightarrow \mathbb{Q}_0^+ \times \mathbb{Q}_0^+$.

I is the **interval function** of $\mathcal{Z}$, $I_1(t)$ and $I_2(t)$ the **earliest firing time of** t ($eft(t)$) and the **latest firing time of** t ($lft(t)$), respectively. It is not difficult to see (cf. [7]) that considering TPNs with $I : T \longrightarrow \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ will not result in a loss of generality. Therefore, only such time functions I will be considered subsequently. Furthermore, conflict is used in the strong sense: two transitions t_1 and t_2 are in **conflict** iff ${}^\bullet t_1 \cap {}^\bullet t_2 \neq \emptyset$. The PN $S(\mathcal{Z})$ is referred to as the **skeleton** of $\mathcal{Z}$.

Within this approach, the definition of a state is of fundamental importance for the ensuing theory. A state is characterized by a marking together with the momentary local time for enabled transitions or the sign ♯ for the disabled transitions.

Definition 2.3. Let $\mathcal{Z} = (P, T, F, V, m_o, I)$ be a TPN and $h : T \longrightarrow \mathbb{R}_0^+ \cup \{\#\}$. $z = (m, h)$ is called a **state** in $\mathcal{Z}$ iff

1. m is a reachable marking in $S(\mathcal{Z})$.
2. $\forall t\ (\ (t \in T \wedge t^- \leq m)\ \longrightarrow\ h(t) \leq lft(t))$.
3. $\forall t\ (\ (t \in T \wedge t^- \not\leq m)\ \longrightarrow\ h(t) = \#)$.

Interpretation of the notion "state" is as follows Within the net, each transition t has a clock $h(t)$. If t is enabled at a marking m, the clock of t $h(t)$ shows the time elapsed since t became most recently enabled. If t is disabled at m, the clock does not work (indicated by $h(t) = \#$). Thus, the vector h which is a vector of clocks is actually a transition marking and the already defined notion "marking" is in fact a place-marking. In the following we call the place-marking m a **p-marking** and the transition-marking h a **t-marking**.

The state $z_o := (m_o, h_o)$ with $h_o(t) := \begin{cases} 0 & \text{iff} \quad t^- \leq m_0 \\ \# & \text{iff} \quad t^- \not\leq m_0 \end{cases}$ is set as the **initial state** of the TPN $\mathcal{Z}$.

Now the dynamic aspects of TPNs – changing from one state into another by firing a transition or by time elapsing – can be introduced:

Definition 2.4. Let $\mathcal{Z} = (P, T, F, V, m_o, I)$ be a TPN, $\hat{t}$ be a transition in T and $z = (m, h)$, $z' = (m', h')$ be two states. Then

1. the transition $\hat{t}$ is **ready** to fire in the state $z = (m, h)$, denoted by $z \xrightarrow{\hat{t}}$, iff
   - (i) $\hat{t}^- \leq m$ and
   - (ii) $eft(\hat{t}) \leq h(\hat{t})$.
2. the state $z = (m, h)$ is **changed** into the state $z' = (m', h')$ **by firing the transition** $\hat{t}$, denoted by $z \xrightarrow{\hat{t}} z'$, iff
   - (i) $\hat{t}$ is ready to fire in the state $z = (m, h)$
   - (ii) $m' = m + \Delta\hat{t}$ and
   - (iii) $\forall t\ \big(\ t \in T \longrightarrow h'(t) := \begin{cases} \# & \text{iff} \quad t^- \not\leq m' \\ h(t) & \text{iff} \quad t^- \leq m \wedge t^- \leq m' \wedge {}^\bullet t \cap {}^\bullet\hat{t} = \emptyset \\ 0 & \text{otherwise} \end{cases}\ \big)$.
3. the state $z = (m, h)$ is **changed** into the state $z' = (m', h')$ **by the time elapsing** $\tau \in \mathbb{R}_0^+$, denoted by $z \xrightarrow{\tau} z'$, iff
   - (i) $m' = m$ and
   - (ii) $\forall t\ (\ t \in T \wedge h(t) \neq \# \longrightarrow h(t) + \tau \leq lft(t)\ )$ i.e. the time elapsing τ is possible, and

(iii) $\forall t\ \big(\ t \in T \longrightarrow h'(t) := \begin{cases} h(t) + \tau & \text{iff} \quad t^- \leq m' \\ \# & \text{iff} \quad t^- \not\leq m' \end{cases}\ \big).$

The state $z = (m, h)$ is called an **integer state** iff $h(t)$ is an integer for each enabled transition t in m.

Definition 2.5. Let $\mathcal{Z} = (P, T, F, V, m_o, I)$ be a TPN.

(a) The state $\hat{z} = (m, h)$ is called **reachable** in $\mathcal{Z}$ (starting at z_0) iff there exist states $z_1, z_1', ..., z_n, z_n'$, transitions $t_1, ..., t_n$ and times $\tau_i \in \mathbb{R}_0^+, i = 1, ..., n+1$ and it holds

$$z_0 \xrightarrow{\tau_1} z_1 \xrightarrow{t_1} z_1' \xrightarrow{\tau_2} z_2 \xrightarrow{t_2} z_2' ... \xrightarrow{\tau_n} z_n \xrightarrow{t_n} z_n' \xrightarrow{\tau_{n+1}} \hat{z}.$$

(b) The set $\mathcal{RS}_{\mathcal{Z}}$ of all reachable states in $\mathcal{Z}$ (starting at z_0) is called the **state space** of $\mathcal{Z}$.

The set of all reachable states in $\mathcal{Z}$, starting at $z \neq z_0$, is denoted by $\mathcal{RS}_{\mathcal{Z}}(z)$. It is easy to see that the set of all reachable p-markings in a TPN $\mathcal{Z}$ is the set $R_{\mathcal{Z}} = \{m \mid (m, h) \in \mathcal{RS}_{\mathcal{Z}}\}$.

The sequence of transitions $(t_1, ..., t_n)$ can fire in $\mathcal{Z}$, starting at z_0, because there is a sequence $(\tau_1, t_1, ..., \tau_n, t_n)$. We denote such a **transition sequence** $\sigma = (t_1, ..., t_n)$ **feasible**. The sequence $\sigma(\tau) = (\tau_1, t_1, ..., \tau_n, t_n)$ which is a concrete execution of σ in $\mathcal{Z}$ is called a **(feasible) run** of σ. It is clear that in a given TPN the state changes are achieved by alternating series of time elapsing and firing. Obviously, for a given run the transition sequence is well defined and for a given transition sequence there are infinitely many runs in general.

At the end of this section we introduce the notion **liveness** for TPNs. Actually, there are four levels of liveness. We consider here only the so called 4-liveness, defined by Lautenbach in [8]. This notion will also be defined in a similar manner to the definition for the classical PNs.

Definition 2.6. Let $\mathcal{Z}$ be a TPN and z a reachable state.

(i) A **transition** t **is live** in the state z iff
$\forall z' \left(z' \in \mathcal{RS}_{\mathcal{Z}}(z) \ \longrightarrow\ \exists z'' \left(m'' \in \mathcal{RS}_{\mathcal{Z}}\left(m'\right) \wedge z'' \xrightarrow{t} \right)\right)$

(ii) A TPN $\mathcal{Z}$ is **live** iff all transitions are live in z_0.

(iii) A TPN $\mathcal{Z}$ is **deadlock-free** iff in each reachable state there is at least one transition t which can (i.e. it is ready to) fire in z.

3. Time-independent liveness

3.1. Preliminaries

In this subsection we introduce some additional notions which are not generally known but which are important for the properties we will study here. Afterwards, we recall some cases for "time-independent liveness" in TPNs, known for more than 15 years, which were the impetus for the present work.

A PN is called **homogeneous** if for each place p holds: $V(p,t) = V(p,t')$ for all $t, t' \in p^\bullet$, i.e. the weights of all arcs from p to its post-transitions are equal. When all of its arc weights are 1's then the PN is an **ordinary** one. An **Extended Free-Choice net (EFC)** is an ordinary PN such that for all $p_1, p_2 \in P$ holds: When $p_1 \cap p_2 \neq \emptyset$ then $p_1^\bullet = p_2^\bullet$. An **Extended Simple net (ES)** is an ordinary PN such that for all $p_1, p_2 \in P$ holds: When $p_1 \cap p_2 \neq \emptyset$ then $p_1^\bullet \subseteq p_2^\bullet$ or $p_2^\bullet \subseteq p_1^\bullet$.

Now we introduce the notions **generalized EFC** and **generalized ES**.

Definition 3.1. A PN $\mathcal{N} = (P, T, F, V, m_o)$ is called a **generalized EFC** iff $\mathcal{N}$ is homogeneous and for all $p_1, p_2 \in P$ holds: When $p_1 \cap p_2 \neq \emptyset$ then $p_1^\bullet = p_2^\bullet$.

Definition 3.2. A PN $\mathcal{N} = (P, T, F, V, m_o)$ is called a **generalized ES** iff $\mathcal{N}$ is homogeneous and for all $p_1, p_2 \in P$ holds: When $p_1 \cap p_2 \neq \emptyset$ then $p_1^\bullet \subseteq p_2^\bullet$ or $p_2^\bullet \subseteq p_1^\bullet$.

The next definition we introduce in order to define "time windows" for transitions in conflict.

Definition 3.3. Let $\mathcal{N} = (P, T, F, V, m_o)$ be a PN and p a place in $\mathcal{N}$. Then $\mathcal{M}in(p) := \max\{eft(t) \mid t \in p^\bullet\}$ and $\mathcal{M}ax(p) := \min\{lft(t) \mid t \in p^\bullet\}$.

It is clear that in a generalized EFC with a place p which $\mathcal{M}in(p)$ is greater than $\mathcal{M}ax(p)$ there is a transition $t \in p^\bullet$ and t can never fire because another transition has to fire before t becomes ready to fire. Thus, the transition t is dead.

3.2. Some structural properies of generalized ES nets

In the following we always consider TPNs $\mathcal{Z} = (P, T, F, V, m_0, I)$ which satisfy the three properties:

(V1) $S(\mathcal{Z})$ is a generalized ES net,

(V2) for every place $p \in P$ holds $\mathcal{M}in(p) \leq \mathcal{M}ax(p)$ and

(V3) for every transition $t \in T$ it holds: $\mathrm{lft}(t) > 0$.

First, we give some definitions and propositions for better understanding the static structure of the generalized ES Nets. Then we analyze the dynamic behaviour of the presumed TPNs.

Definition 3.4. A place $p \in P$ is of first **degree** iff

$$\forall q \in P : p^\bullet \cap q^\bullet \neq \emptyset \Longrightarrow p^\bullet \supseteq q^\bullet.$$

A place $p \in P$ is of n-th degree iff $p^\bullet \neq \emptyset$ and

$$n = \max_{q \in P} \{k \mid q^\bullet \supsetneq p^\bullet \wedge q \text{ is of } k\text{-th degree}\} + 1.$$

Theorem 1. Definition 3.4 is well-defined.

[0]Also known as asymmetric choice net (cp. [11]).

Proof: We have to show that a place can never be of two different degrees.

According to definition 3.4, a place is of first degree when either it has no post-transitions or there is no place with a proper superset of post-transitions. Therefore, no place of first degree can be also of higher degree.

Let $p \in P$ be a place of m-th and n-th degree, whereas m and n are greater than 1. Then w.r.t. definition 3.4 it holds:

$$\exists q_m \in P : q_m \text{ is of } (m-1)\text{-th degree} \wedge {q_m}^\bullet \supsetneq p^\bullet \text{ and } \nexists r \in P : {q_m}^\bullet \supsetneq r^\bullet \supsetneq p^\bullet \text{ and}$$
$$\exists q_n \in P : q_n \text{ is of } (n-1)\text{-th degree} \wedge {q_n}^\bullet \supsetneq p^\bullet \text{ and } \nexists r \in P : {q_n}^\bullet \supsetneq r^\bullet \supsetneq p^\bullet. \quad (1)$$

Then the inclusion $\emptyset \neq p^\bullet \subseteq {q_m}^\bullet \cap {q_n}^\bullet$ follows immediately and because of property (V1) and w.l.o.g. also ${q_m}^\bullet \subseteq {q_n}^\bullet$. Since $p^\bullet \subsetneq {q_m}^\bullet \subsetneq {q_n}^\bullet$ and (1) hold the inclusion can be no proper. This leads to

$${q_m}^\bullet = {q_n}^\bullet \Longrightarrow \forall r \in P : (r^\bullet \subsetneq {q_m}^\bullet \Leftrightarrow r^\bullet \subsetneq {q_n}^\bullet) \Longrightarrow$$
$$m = \max_{r \in P} \{k \mid r^\bullet \supsetneq {q_m}^\bullet \wedge r \text{ is of } k\text{-th degree}\} = \max_{r \in P} \{k \mid r^\bullet \supsetneq {q_n}^\bullet \wedge r \text{ is of } k\text{-th degree}\} = n \square$$

Lemma 3.1. Two places $p, q \in P$ of same degree with at least one common post-transition have the same set of post-transitions.

Proof: Let p and q be two places with $p^\bullet \cap q^\bullet \neq \emptyset$. Because of (V1) the subset relation $p^\bullet \subseteq q^\bullet$ or $q^\bullet \subseteq p^\bullet$ is true. If the inclusion would be a proper, meaning w.l.o.g $q^\bullet \supsetneq p^\bullet$, then w.r.t. definition 3.4 p would be of higher degree than q. Thus, the inclusion cannot be proper and therefore $p^\bullet = q^\bullet$. □

Lemma 3.2. Let a transition $t \in T$ have a pre-place of n-th degree. Then for every i with $1 \leq i \leq n$ t has a pre-place of i-th degree.

Proof: The proof will be done by induction. The initial step is trivial.

Inductive step: Let a place $p \in {}^\bullet t$ be of $(n+1)$-th degree. Then according to definition 3.4 there is a place $p_n \in P$ of n-th degree with $t \in p^\bullet \subsetneq p_n^\bullet$. This implies $p_n \in {}^\bullet t$. □

Definition 3.5. A **transition** $t \in T$ **is of** n**-th degree** iff

$$n = \max_{p \in {}^\bullet t} \{k \mid p \text{ is of } k\text{-th degree}\}.$$

Lemma 3.3. For every two transitions $s, t \in T$ and a common pre-place $p \in {}^\bullet s \cap {}^\bullet t$ of k-th degree holds

$$\forall i \in \mathbb{N}\ \forall q \in {}^\bullet t : 1 \leq i \leq k \wedge q \text{ is of } i\text{-th degree} \Longrightarrow q \in {}^\bullet s.$$

Proof: Let $q \in {}^\bullet t$ be a pre-place of i-th degree with $1 \leq i \leq k$. Then q is a pre-place of s because of

$$\{s, t\} \subseteq p^\bullet \subseteq q^\bullet \Longrightarrow q \in {}^\bullet s.$$
□

Definition 3.6. 1. Two transitions $s, t \in T$ are in **static conflict** (short: $s \between t$) iff they have a common pre-place, i.e.

$$s \between t :\Longleftrightarrow {}^\bullet s \cap {}^\bullet t \neq \emptyset.$$

2. A transition $s \in T$ is **dominating** a transition $t \in T$ w.r.t. a set of places $M \subseteq P$ (short: $t <_M s$) iff s and t are in static conflict and the set of pre-places of s relative to M is a proper subset of the set of pre-places of t relative to M i.e.

$$t <_M s :\Longleftrightarrow s \between t \wedge {}^\bullet s \cap M \subsetneq {}^\bullet t \cap M.$$

3. Two transitions $s, t \in T$ are **equivalent** w.r.t. a set of places $M \subseteq P$ (short: $s \sim_M t$) iff they have the same pre-places relative to M, i.e.

$$s \sim_M t :\Longleftrightarrow {}^\bullet s \cap M = {}^\bullet t \cap M.$$

It is clear that each transition $t \in T$ is always in conflict with itself.

Lemma 3.4. The relation $\between$ is an equivalence relation over T.

Proof: Reflexivity and symmetry follow immediately from definition 3.6. Let $r, s, t \in T$ be transitions with $r \between s$ and $s \between t$. Then there exist common pre-places $\tilde{p}$ and $\tilde{q}$ with $\tilde{p} \in {}^\bullet r \cap {}^\bullet s$ of m-th degree and $\tilde{q} \in {}^\bullet s \cap {}^\bullet t$ of n-th degree. Then, due to lemmas 3.2 and 3.3 the following holds:

$$\exists p \in {}^\bullet r \cap {}^\bullet s : p \text{ is of 1-th degree and } \exists q \in {}^\bullet s \cap {}^\bullet t : q \text{ is of 1-th degree}.$$

Now, applying lemma 3.1 we obtain

$$s \in p^\bullet \cap q^\bullet \Longrightarrow \{r, s, t\} \subseteq p^\bullet = q^\bullet \Longrightarrow \emptyset \neq \{p, q\} \subseteq {}^\bullet r \cap {}^\bullet t \Longrightarrow r \between t.$$

□

Lemma 3.5. The relation $\sim_M$ is an equivalence relation over T for every subset $M \subseteq P$.

Proof: The relation $\sim_M$ is an equivalence relation, since the equality ($=$) of sets is an equivalence relation. □

Lemma 3.6. The relation $<_M$ is transitive for every $M \subseteq P$.

Proof: This holds because of the transitivity of the subset relation ($\subseteq$). □

Lemma 3.7. For every two transitions $s, t \in T$ which are in static conflict it holds: $t \sim_{{}^\bullet t} s \vee t <_{{}^\bullet t} s$.

Proof: Let s and t be in static conflict. Hence, it holds: ${}^\bullet s \cap {}^\bullet t \neq \emptyset$ and therefore also $\emptyset \neq {}^\bullet s \cap {}^\bullet t \subseteq {}^\bullet t$. There are two cases to be considered:
Case 1: ${}^\bullet s \cap {}^\bullet t = {}^\bullet t$. Then according to the definition 3.6 (subitem (3)) it follows: $t \sim_{{}^\bullet t} s$.
Case 2: ${}^\bullet s \cap {}^\bullet t \subsetneq {}^\bullet t$. Then according to the definition 3.6 (subitem (2)) it follows: $t <_{{}^\bullet t} s$. □

3.3. When the skeleton is live, then the TPN is live as well

Definition 3.7. A **place** $p \in P$ **is live** in a state z iff p has a pre-transition $t \in {}^{\bullet}p$ which is live in z, i.e.

$$p \text{ is live in } z :\Longleftrightarrow \exists t \in {}^{\bullet}p : t \text{ is live in } z.$$

A place $p \in P$ is dead in the state z iff all of its pre-transitions are dead in z, i.e.

$$p \text{ is dead in } z :\Longleftrightarrow \forall t \in {}^{\bullet}p : t \text{ is dead in } z.$$

Lemma 3.8. Let $z \in RS_{\mathcal{Z}}$ be a reachable state. Then there is a state $z' \in RS_{\mathcal{Z}}(z)$ such that every transition and, therefore, every place is either dead or live in that state, i.e.

$$\forall t \in T : t \text{ is dead in } z' \vee t \text{ is live in } z' \qquad \text{and} \tag{2}$$

$$\forall p \in P : p \text{ is dead in } z' \vee p \text{ is live in } z'. \tag{3}$$

Proof: Let $z \in RS_{\mathcal{Z}}$ be a reachable state, $t \in T$ an arbitrary transition and let t be not live in z. Then there exists a state $z^1 \in RS_{\mathcal{Z}}(z)$ such that t is dead in z^1. Beacause of the finiteness of the set T there is a $n \in \mathbb{N}$ such that there is no transition which is not live in z^n. This means that every transition is either dead or live in z^n. Hence, (2) is true for $z' := z^n$. Property (3) follows immediately from (2) because of definition 3.7. □

Definition 3.8. Let $z \in RS_{\mathcal{Z}}$ be a reachable state, $\sigma(\tau) = \tau_0 t_1 \tau_1 \dots t_n \tau_n$ be a feasible run with:

$$\text{if } \tau_n = 0 \text{ then } n = 0 \tag{4}$$

and

$$z \xrightarrow{\tau_0} z_0^1 \xrightarrow{t_1} z_1^0 \xrightarrow{\tau_1} z_1^1 \xrightarrow{t_2} z_2^0 \xrightarrow{\tau_2} z_2^1 \xrightarrow{t_3} \dots \xrightarrow{\tau_{n-1}} z_{n-1}^1 \xrightarrow{t_n} z_n^0 \xrightarrow{\tau_n} z_n^1,$$

where for the states $z_i^1 = (m_i^1, h_i^1)$ holds

$$\forall i \in \mathbb{N} : 0 \le i < n \Longrightarrow h_i^1(t_{i+1}) = \text{lft}(t_{i+1}).$$

Then $\sigma(\tau)$ is called a **forced run** starting from z. Furthermore, $\sigma(\tau)$ is a forced run, passing the time e iff

$$\sum_{i=0}^{n} \tau_i = e.$$

Property (4) claims that the forced run consists of a minimal number of transitions in order to pass the time $e \ge 0$.

Lemma 3.9. Let $\mathcal{Z} = (T, P, F, V, m, I)$ be a deadlock-free Time Petri net such that only the restriction (V3) holds. Then for every reachable state $z \in RS_{\mathcal{Z}}$ and every time $e \ge 0$ there is a feasible run $\sigma(\tau)$ with $\sigma = t_1 \dots t_n$, $\tau = \tau_0 \dots \tau_n$ and starting at z and it holds $\sum_{i=0}^{n} \tau_i = e$. Even more, it is possible to find a forced run starting from $z \in RS_{\mathcal{Z}}$ passing the time $e \ge 0$.

Proof: Let $\hat{\tau} := \min\limits_{t\in T}\{\text{lft}(t)\}$ be the smallest lft in $\mathcal{Z}$ and $z_1 = (m_1, h_1) \in \mathit{RS}_{\mathcal{Z}}$ be a reachable state.

Successively, transitions reaching their latest firing time will fire. After that, as much time as possible will be passed. If more than $|T|$ transitions will have been fired, then at least $\hat{\tau}$ time will have passed. This process will be repeated as much as needed, thus the proof will be done by nested recursion. The counter variable for the outer recursion will be i and the one for the inner recursion will be j.

For $i \geq 1$ successively set $M_i^0 := \{t \in T \mid h_i(t) = \text{lft}(t)\}$ and let $t_i^0 \in M_i^0$ be an arbitrary transition whose clock reached its latest firing time. Then set $\sigma_i^0 := t_i^0$ and let $z_i^1 \in \mathit{RS}_{\mathcal{Z}}(z_i)$ be the state defined by

$$z_i \xrightarrow{\sigma_i^0} z_i^1 = (m_i^1, h_i^1).$$

This describes the outer recursion.

With the inner recursion a state $\tilde{z}_i := z_i^{n_i}$ will be constructed such that no clock of an enabled transition is at its latest firing time. So, for $j \geq 1$ set

$$M_i^j := \left\{t \in T \mid h_i^j(t) = \text{lft}(t)\right\} \subsetneq M_i^{j-1} \tag{5}$$

and let $t_i^j \in M_i^j$ be a transition whose clock is at its latest firing time. Furthermore, set $\sigma_i^j := \sigma_i^{j-1} t_i^j$ and let z_i^{j+1} be the state defined by $z_i^j \xrightarrow{t_i^j} z_i^{j+1}$. Obviously, the sets M_i^j will be smaller with each step, thus $M_i^{n_i} = \emptyset$ for a $n_i \in \mathbb{N}$. That is the way a state $\tilde{z}_i = (\tilde{m}_i, \tilde{h}_i) := z_i^{n_i}$ will be reached after firing

$$\sigma_i := \sigma_i^{n_i} = t_i^0 \dots t_i^{n_i-1}. \tag{6}$$

Now, a certain amount of time can be passed. Set $\tau_0 := 0$ and

$$\tau_i := \min\left\{ e - \sum_{k=0}^{i-1} \tau_k,\ \max_{\substack{t\,\in\,T \\ \tilde{h}_i(t)\,\neq\,\#}} \left\{\text{lft}(t) - \tilde{h}_i(t)\right\} \right\} \tag{7}$$

and let $z_{i+1} = (m_{i+1}, h_{i+1})$ be the state defined by $z_i \xrightarrow{\sigma_i} \tilde{z}_i \xrightarrow{\tau_i} z_{i+1}$.

Because $\mathcal{Z}$ is deadlock-free there is always a transition $t \in T$ such that $\tilde{h}_i(t) \neq \#$ holds. Hence, this definition is well-defined. If $\tau_i = 0$ then $\sum_{k=0}^{i-1} \tau_k = e$ is true because of (7) and because of the basic property ($\forall t \in T : \tilde{h}_i(t) < \text{lft}(t) \vee \tilde{h}_i(t) = \#$). Otherwise i will be incremented and the process is going to be continued. Now, $\sigma := \sigma_1\tau_1\sigma_2\tau_2 \dots \sigma_{i-1}\tau_{i-1}$ is a feasible and forced run starting from z_1 passing the time e.

This process brings a finite feasible run starting from z which can be seen by the following. There are n_i transitions firing in every run $\sigma_i\tau_i$. After that the time τ_i is passed. (5) and (6) imply $n_i \leq |T|$ and $\forall i > 1 : 1 \leq n_i$, thus

$$\exists \tilde{i} \in \mathbb{N} : |T| < \sum_{k=1}^{\tilde{i}} n_k \leq 2|T|. \tag{8}$$

The left unequation of (8) implies that there is a transition $\hat{t} \in \sigma_1 \dots \sigma_{\tilde{i}}$ which fires at least twice in that run, thus its clock has been reset and the second time it fired at its latest firing time. Then

$$\sum_{k=1}^{\tilde{i}-1} \tau_k \geq \text{lft}(\hat{t}) \geq \hat{\tau}$$

holds. The right unequation of (8) implies that the run is finite. By repeating that process $n := \left\lceil \frac{\tau}{\hat{\tau}} \right\rceil + 1$ times, the number of transitions which fired is not greater then $2\left(\left\lceil \frac{\tau}{\hat{\tau}} \right\rceil + 1\right)|T|$. Furthermore, there is at least the time τ passed, since

$$\tau \le n\hat{\tau} = \hat{\tau}\left(\left\lceil \frac{\tau}{\hat{\tau}} \right\rceil + 1\right) \Longleftrightarrow \frac{\tau}{\hat{\tau}} - 1 \le \left\lceil \frac{\tau}{\hat{\tau}} \right\rceil .$$

□

Lemma 3.10. Let $z^0 = (m^0, h^0) \in RS_{\mathcal{Z}}$ be a reachable state in $\mathcal{Z}$, $t^* \in T$ be an arbitrary transition and ${}^{\bullet}t^* = \{p_1, \ldots, p_n\}$ be the set of the pre-places of t^* "ordered/indexed" by descending degree, i.e.

$$\forall i \in \mathbb{N}\ \forall j \in \mathbb{N} : 1 \le i \le j \le n \wedge p_i \text{ is of } c\text{-th degree } \wedge p_j \text{ is of } d\text{-th degree } \Longrightarrow c \ge d.$$

Furthermore, let $M_0 = \emptyset$ and $M_i := \{p_1, \ldots, p_i\}$ be for each $1 \le i \le n$.

If there is an $\tilde{i}$ with $0 \le \tilde{i} < n$ such that

$$\forall p \in {}^{\bullet}t^* \setminus M_{\tilde{i}} : p \text{ is live in } z^0, \tag{9}$$

$$\forall p \in M_{\tilde{i}} : m^0(p) \ge V(p, t^*) \qquad \text{and} \tag{10}$$

$$\forall t \in T : t \between t^* \Longrightarrow h^0(t) = 0 \vee h^0(t) = \#, \tag{11}$$

then there is a state $z^{n+1} \in RS_{\mathcal{Z}}(z^0)$ and t^* is ready to fire in state z^{n+1}.

Proof: The proof is done by induction.

Base: At the beginning set $i = \tilde{i}$ and $z_0^{i+1} = z^0$. The assumptions for the next step in the recursion are made of (10) and (11).

Step: $(i-1) \longrightarrow i$**:** First set $j = 1$. In every case of the following case dfferentiation where t^* is not ready to fire the assumptions

$$\forall p \in M_{i-1} : m^i_{j+1}(p) \ge V(p, t^*) \text{ and } m^i_{j+1}(p_i) > m^i_j(p_i) \text{ and} \tag{12}$$

$$\forall t \in T : t \between t^* \Longrightarrow h^i_j(t) = 0 \vee h^i_j(t) = \# \tag{13}$$

will be prooved for every reached pair (i, j). This especially implies (10) and (11). The place p_i is live in z^i_{j-1} since $i > \tilde{i}$ and (9) holds, meaning $\exists t_i \in {}^{\bullet}p_i : t_i$ is live in z^i_{j-1}, wherefore there is a feasible run starting from z^i_{j-1} and firing t_i at the end. This increases the number of tokens on p_i.

$$\exists \hat{\sigma}^i_j : z^i_{j-1} \xrightarrow{\hat{\sigma}^i_j} \hat{z}^i_j = (\hat{m}^i_j, \hat{h}^i_j) \wedge \hat{m}^i_j(p_i) > m^i_{j-1}(p_i).$$

Case 1: $\forall t \in \hat{\sigma}^i_j : {}^{\bullet}t \cap M_i \neq \emptyset$, meaning the number of tokens did not decrease on places of M_i. Then set $\tilde{z}^i_j = (\tilde{m}^i_j, \tilde{h}^i_j) := \hat{z}^i_j$ and $\tilde{\sigma}^i_j := \hat{\sigma}^i_j$.

Case 2: $\exists \tilde{t} \in \hat{\sigma}^i_j : {}^{\bullet}\tilde{t} \cap M_i \neq \emptyset$. In this case a transition with a pre-place of M_i must have been firing during $\hat{\sigma}^i_j$. Set $\hat{\sigma}^i_j = {}^0\hat{\tau}^i_j\ {}^1\hat{t}^i_j\ {}^1\hat{\tau}^i_j \ldots {}^b\hat{t}^i_j\ {}^b\hat{\tau}^i_j$ for a $b \in \mathbb{N}$. Then

$$\exists a \in \mathbb{N} : 1 \le a \le b \Longrightarrow {}^{\bullet}({}^a\hat{t}^i_j) \cap M_i \neq \emptyset \wedge \left(\forall c \in \mathbb{N} : 1 \le c < a \Longrightarrow {}^{\bullet}({}^c\hat{t}^i_j) \cap M_i = \emptyset\right),$$

i.e. ${}^a\hat{t}^i_j$ is the first transition of $\hat{\sigma}^i_j$ which has a pre-place of M_i. W.r.t. lemma 3.3 this transition has also p_i as a pre-place. Define $\tilde{\sigma}^i_j := {}^0\hat{\tau}^i_j\ {}^1\hat{t}^i_j\ {}^1\hat{\tau}^i_j \ldots {}^{a-1}\hat{t}^i_j$ and the state $\tilde{z}^i_j = (\tilde{m}^i_j, \tilde{h}^i_j)$ by $z^i_{j_1} \xrightarrow{\tilde{\sigma}^i_j} \tilde{z}^i_j$.

In both cases a state $\tilde{z}^i_j$ is reached fulfilling (12) (substitute m^i_{j+1} by $\tilde{m}^i_j$).

The following case differentiation constructs a state $z^i_j \in RS_{\mathcal{Z}}(\tilde{z}^i_j)$ which fulfills (13), as well.

Case 1: In $\tilde{z}^i_j$ the assumption (13) is already fulfilled. Then set $z^i_j = (m^i_j, h^i_j) := \tilde{z}^i_j$.

Case 2: $\exists \tilde{t} \in T : \tilde{t} \between t^* \wedge \hat{h}^i_j(\tilde{t}) > 0$. Due to Lemma 3.9 there exists a forced run $\bar{\sigma}^i_j$ with

$$\tilde{z}^i_j \xrightarrow{\bar{\sigma}^i_j} \bar{z}^i_j = (\bar{m}^i_j, \bar{h}^i_j).$$

passing the time $\mathrm{lft}(\tilde{t}) - \tilde{h}^i_j(\tilde{t})$ starting from $\tilde{z}^i_j$. Remark that all transitions of $\bar{\sigma}^i_j$ fire at their latest firing time during.

Case 2.1: $\bar{h}^i_j(\tilde{t}) = \mathrm{lft}(\tilde{t})$. This means that the transition $\tilde{t}$ which impedes the asumptions (13) for the next firing is ready to fire in $\bar{z}^i_j$. $\forall t \in \bar{\sigma}^i_j : {}^\bullet t \cap M_i = \emptyset$ holds since $\bar{\sigma}^i_j$ is a forced run starting from $\tilde{z}^i_j$. The firing of $\tilde{t}$ could decrease the number of tokens of a place of M_i.

Case 2.1.1: ${}^\bullet\tilde{t} \cap M_i \neq \emptyset$. For all states ${}^s\breve{z}^i_j = ({}^s\breve{m}^i_j, {}^s\breve{h}^i_j)$ which has been reached during $\bar{\sigma}^i_j$ the assumption

$$(t^*)^-\Big|_{M_i} \leq {}^s\breve{m}^i_j\Big|_{M_i}$$

holds. This face, lemma 3.3 and V2 brings $\tilde{t}^- \leq {}^s\breve{m}^i_j \implies (t^*)^- \leq {}^s\breve{m}^i_j$, wherefore $\bar{h}^i_j(\tilde{t}) = \bar{h}^i_j(t^*)$. Finally t^* is ready to fire in $\bar{z}^i_j$ because of $\mathrm{eft}(t^*) \leq \mathrm{lft}(\tilde{t}) = \bar{h}^i_j(\tilde{t})$ which holds because of V3. In this case the proof is done.

Case 2.1.2: ${}^\bullet\tilde{t} \cap M_i = \emptyset$. Then $\forall 1 \leq k \leq i : p_k \notin {}^\bullet\tilde{t}$ holds. The assumptions (12) and (13) for the next step of the recursion are fulfilled with $\bar{z}^i_j \xrightarrow{\tilde{t}} z^i_j$. In this case set $\sigma^i_j := \tilde{\sigma}^i_j\bar{\sigma}^i_j\tilde{t}$.

Case 2.2: $\bar{h}^i_j(\tilde{t}) < \mathrm{lft}(\tilde{t}) \vee \bar{h}^i_j(\tilde{t}) = \#$. In this case a transition which is in static conflict with $\tilde{t}$ fired during $\bar{\sigma}^i_j$, meaning $\exists t \in \bar{\sigma}^i_j : t \between \tilde{t}$. Set $\bar{\sigma}^i_j = \bar{\tau}_0\bar{t}_1\bar{\tau}_1 \ldots \bar{t}_r\bar{\tau}_r$. Then

$$\exists \bar{t}_a \in \bar{\sigma}^i_j : \bar{t}_a \between t^* \wedge \forall 1 \leq b < a : {}^\bullet\bar{t}_b \cap {}^\bullet t^* = \emptyset \tag{14}$$

holds because of lemma 3.4. Then define $\dot{\sigma}^i_j := \bar{\tau}_0\bar{t}_1 \ldots \bar{\tau}_{a-1}$ and $\dot{z}^i_j = (\dot{m}^i_j, \dot{h}^i_j)$ by $\tilde{z}^i_j \xrightarrow{\dot{\sigma}^i_j} \dot{z}^i_j$.

Case 2.2.1: ${}^\bullet\bar{t}_a \cap M_i \neq \emptyset$. For all states ${}^s\breve{z}^i_j = ({}^s\breve{m}^i_j, {}^s\breve{h}^i_j)$ reached during $\dot{\sigma}^i_j$

$$(t^*)^-\Big|_{M_i} \leq^s \breve{m}^i_j\Big|_{M_i}$$

holds analogously to case 2.1.1. In addition lemma 3.3 and V2 holds wherefore $\bar{t}_a^- \leq^s \breve{m}^i_j \implies (t^*)^- \leq^s \breve{m}^i_j$ holds for these states. Set $\dot{h}^i_j(\bar{t}_a) = \dot{h}^i_j(t^*)$. Finally, t^* is ready to fire in $\dot{z}^i_j$ because $\mathrm{eft}(t^*) \leq \mathrm{lft}(\dot{t}_a) = \dot{h}^i_j(\bar{t}_a)$ holds. In this case the proof is done.

Case 2.2.2: ${}^\bullet\bar{t}_a \cap M_i = \emptyset$. Define the state z^i_j by $\dot{z}^i_j \xrightarrow{\bar{t}_a} z^i_j$. The assumptions (12) and (13) are then fulfilled in z^i_j since (14) holds. Then set $\sigma^i_j := \tilde{\sigma}^i_j\dot{\sigma}^i_j\bar{t}_a$.

By successively repeating this process for $j > 1$ until $m^i_j(p_i) \geq V(p_i, t^*)$ holds, a feasible run $\sigma^i = \sigma^i_1 \ldots \sigma^i_{j_1}$ starting from z^i_0 is reached or a state is reached during that process in which t^* is ready to fire. This process will finish after finite steps since the number of tokens on p_i raise for increasing j. Let z_0^{i+1} be the state defined by

$$z_0^i \xrightarrow{\sigma^i} z_0^{i+1}.$$

These steps will be repeated for $i = (\tilde{i}+1), \dots, n$. Then t^* fires or a state z^{n+1} is reached in which t^* is enabled and

$$\forall t \in T : t \between t^* \Longrightarrow h^{n+1}(t) = 0 \vee h^{n+1}(t) = \# \tag{15}$$

holds. Let σ^f be a forced run passing the time eft(t^*) starting from z^{n+1}. Then $\forall t \in \sigma^f : {}^\bullet t \cap {}^\bullet t^* = \emptyset$ holds since (V3) and (15) is fulfilled. Let $z^f = (m^f, h^f)$ be the state defined by

$$z^{n+1} \xrightarrow{\sigma^f} z^f.$$

Then $(t^*)^- \leq m^f$ and eft$(t^*) = h^f(t^*)$ holds, i.e. t^* is ready to fire in z^f. □

Lemma 3.11. Let $z = (m, h) \in RS_{\mathcal{Z}}$ be a reachable state in $\mathcal{Z}$ and $t^* \in T$ be an arbitrary transition. Then if each pre-place p of t^* is live than t^* is live, as well.

Proof: Let $z^0 = (m^0, h^0) \in RS_{\mathcal{Z}}(z)$ be an arbitrary state reachable from z. In the following a state z^2 will be constructed which fulfill the conditions of lemma 3.10.

Case 1: If the condition (11) of lemma 3.10 is already fulfilled then set $z^2 := z^0$.

Case 2: $\exists \tilde{t} \in T : \tilde{t} \between t^* \wedge h^0(\tilde{t}) > 0$. Let σ^1 be a forced run passing the time lft$(\tilde{t}) - h^0(\tilde{t})$ starting from z^0. Define the state z^1 by

$$z^0 \xrightarrow{\sigma^1} z^1 = (m^1, h^1).$$

Case 2.1: $h^1(\tilde{t}) =$ lft$(\tilde{t})$. The let z^2 be the state defined by $z^0 \xrightarrow{\sigma^1} z^1 \xrightarrow{\tilde{t}} z^2 = (m^2, h^2)$. In this state the condition $\forall t \in T : t \between t^* \Longrightarrow h^2(t) = 0 \vee h^2(t) = \#$ holds.

Case 2.2: $h^1(\tilde{t}) <$ lft$(\tilde{t}) \vee h^1(\tilde{t}) = \#$. Define $\sigma^1 := \tau_0 t_1 \tau_1 \dots t_r \tau_r$. Then

$$\exists t_a \in \sigma^1 : t_a \between t^* \wedge \forall 1 \leq b \leq a : {}^\bullet t_b \cap {}^\bullet t^* = \emptyset$$

must hold. In this case let z^2 be the state defined by $z^0 \xrightarrow{\tau_0 t_1 \dots t_a} z^2$. In this state the assumption $\forall t \in T : t \between t^* \Longrightarrow h^2(t) = 0 \vee h^2(t) = \#$ holds.

In every case z^2 fulfills the assumptions for lemma 3.10 with $\tilde{i} = 0$, wherefore there is a following state $z^3 \in RS_{\mathcal{Z}}(z^2)$ in which t^* is ready to fire. □

Lemma 3.12. Let $z \in RS_{\mathcal{Z}}$ be a reachable state in $\mathcal{Z}$. Then iff $t \in T$ is dead in z the following holds:

$$\exists p \in {}^\bullet t \; \exists z' \in RS_{\mathcal{Z}}(z) : p \text{ is dead in } z'.$$

Proof: By lemma 3.8 there is a following state $z' \in RS_{\mathcal{Z}}(z)$ in which all transitions and places are either dead or live, meaning

$$\forall t \in T : t \text{ is dead in } z' \vee t \text{ is live in } z' \text{ and}$$
$$\forall p \in P : p \text{ is dead in } z' \vee p \text{ is live in } z'.$$

If all $p \in {}^\bullet t$ would be live in z' then, by lemma 3.11, t would be live in z' in contradiction to the assumptions. □

Lemma 3.13. Let $z = (m, h) \in RS_{\mathcal{Z}}$ be a reachable state and $t^* \in T$ be a transition with the properties: Every pre-place of t^* is dead or live in z and at least one pre-place of t^* is dead in z. Then there is a state $\tilde{z} = (\tilde{m}, \tilde{h}) \in RS_{\mathcal{Z}}(z)$ and it holds:

$$\exists p \in {}^\bullet t^* : p \text{ is dead in } \tilde{z} \wedge \tilde{m}(p) < V(p, t^*).$$

Proof: Let ${}^\bullet t^* = \{p_1, \ldots, p_n\}$ be the set of the pre-places of t^* indexed by descending degree, i.e.

$$\forall i \in \mathbb{N}\ \forall j \in \mathbb{N} : 1 \leq i \leq j \leq n \wedge p_i \text{ is of } c\text{-th degree } \wedge p_j \text{ is of } d\text{-ter degree } \Longrightarrow c \geq d.$$

Furthermore, let

$$p_m \in {}^\bullet t^* : p_m \text{ is dead in } z \wedge \forall m < i \leq n : p_i \text{ is live in } z$$

be the last (w.r.t. the indices) pre-place of t^* which is dead in z, let

$$L_1 := \{p_i \in {}^\bullet t^* \mid 1 \leq i < m \wedge p_i \text{ is live in } z\}$$

be the set of pre-places which are live in z and have at least the same degree as p_m and eventually let

$$L_2 := \{p_i \in {}^\bullet t^* \mid m < i \leq n \wedge p_i \text{ is live in } z\}$$

be the set of pre-places which are live in z and have at most the same degree as p_m. Furthermore, w.l.o.g. let $\forall p \in {}^\bullet t^* : p$ is dead in $z \Longrightarrow m(p) \geq V(p, t^*)$. Otherwise, the proof would already be done.

Due to the definitions of L_1 and L_2 there is a transition $t_i \in {}^\bullet p_i$ which is live in z for every $p_i \in L_1 \cup L_2$. Hence, a feasible run σ_1 with $z \xrightarrow{\sigma_1} z_1 = (m_1, h_1)$ starting from z exists such that the pre-transition t_i of $p_i \in L_1$ which is live in z occurs at least $V(p_i, t^*)$ times in σ_1. Following, the number of tokens on the places $p_i \in L_1$ increased at least $V(p_i, t^*)$ times during σ^1.

Case 1: $\exists p_i \in L_1 : m_1(p_i) < V(p_i, t^*)$. Then $\exists \tilde{t} \in \sigma_1 : p_i \in {}^\bullet \tilde{t}$ must hold. Due to lemma 3.3 the place p_m is also a pre-place of $\tilde{t}$. Thus, the number of tokens on p_m in z_1 is less than the number of tokens on p_m in z.

Case 2: $\forall p_i \in L_1 : m_1(p_i) \geq V(p_i, t^*)$.

Case 2.1: $\forall t \in T : t \between t^* \Longrightarrow h^1(t) = 0 \vee h^1(t) = \#$. In this case the conditions for lemma 3.10 are fulfilled with L_1 as $M_{\tilde{i}}$. Thus, there is a state $z^2 \in RS_{\mathcal{Z}}(z^1)$ in which t^* is ready to fire. By firing t^* in z^2 the number of tokens on p_m decreases.

Case 2.2: $\exists \tilde{t} \in T : \tilde{t} \between t^* \wedge h^1(\tilde{t}) > 0$. Then the forced run σ_2, started at z^1, passing the time $\text{lft}(\tilde{t}) - h^1(\tilde{t})$ achieves a state $z^2 = (m^2, h^2) \in RS_{\mathcal{Z}}(z^1)$, i.e. $z^1 \xrightarrow{\sigma_2} z^2$

Case 2.2.1 $h^2(\tilde{t}) = \text{lft}(\tilde{t})$. Then $\tilde{t}$ is ready to fire in z^2.

Case 2.2.1.1 ${}^\bullet \tilde{t} \cap L_1 = \emptyset$. In this case the conditions of lemma 3.10 are fulfilled with L_1 as $M_{\tilde{i}}$ in the state following z^2 by firing $\tilde{t}$. Hence, there is a following state z^3 such that t^* is ready to fire in z^3.

Case 2.2.1.2 ${}^\bullet \tilde{t} \cap L_1 \neq \emptyset$. Due to lemma 3.3, p_m is also a pre-place of $\tilde{t}$, thus the number of tokens on p_m decrease by firing $\tilde{t}$ in z^2.

Case 2.2.2 $h^2(\tilde{t}) \neq \text{lft}(\tilde{t})$. In this case a transition $\hat{t} \in \sigma_2$, which is in static conflict with $\tilde{t}$ had to fire.

Case 2.2.2.1 $^\bullet\hat{t} \cap L_1 = \emptyset$. In this case the conditions of lemma 3.10 are fulfilled with L_1 as $M_{\tilde{i}}$. Thus there is a state $z^3 \in RS_{\mathcal{Z}}(z^2)$ in which t^* is ready to fire. After the firing of t^* in z^3 the number of tokens on p_m decrease.

Case 2.2.2.2 $^\bullet\hat{t} \cap L_1 \neq \emptyset$. Due to lemma 3.3, p_m is also a pre-place of $\hat{t}$ and therefore the number of tokens on p_m in z_2 is less than the number of tokens on p_m in z_1.

Altogether the number of tokens on p_m decreased. Since p_m is dead in z, the number of tokens on p_m cannot be increased in a following state. Hence, by repeating this process a state $\tilde{z} = (\tilde{m}, \tilde{h})$ will be reached after finite steps and $\tilde{m}(p_m) < V(p_m, t^*)$ holds. □

Lemma 3.14. Let $z \in RS_{\mathcal{Z}}$ be a reachable state and $t \in T$ be a transition which is dead in z. Then there is a state $z' = (m', h') \in RS_{\mathcal{Z}}(z)$ such that the following holds:

$$\exists p \in {}^\bullet t : p \text{ is dead in } z' \wedge m'(p) < V(p, t).$$

Proof: According to lemma 3.8 there is a state $z'' \in RS_{\mathcal{Z}}(z)$ such that every transition and every place is either dead or live in that state. Especially, t is dead in z''. Due to lemma 3.12 it holds:

$$\exists z''' \in RS_{\mathcal{Z}}(z'') \subseteq RS_{\mathcal{Z}}(z)\ \exists p \in {}^\bullet t : p \text{ is dead in } z'''.$$

Then because of lemma 3.13 it follows:

$$\exists z' = (m', h') \in RS_{\mathcal{Z}}(z''') \subseteq RS_{\mathcal{Z}}(z)\ \exists p \in {}^\bullet t : p \text{ is dead in } z' \wedge m'(p) < V(p, t).$$

□

Lemma 3.15. Let $z \in \mathcal{Z}$ be a reachable state. If all transitions are either dead or live in z, i.e.

$$\forall t \in T : t \text{ is dead in } z \vee t \text{ is live in } z,$$

then there is a state $z' \in RS_{\mathcal{Z}}(z)$ such that every transition which is dead in that state has a pre-place which has not enough tokens for the purpose of enabling this transition, i.e.

$$\forall t \in T : t \text{ is dead in } z' \Longrightarrow \exists p \in {}^\bullet t : p \text{ is dead in } z' \wedge m'(p) < V(p, t).$$

Proof: Let $z^0 = (m^0, h^0) := z$ and $\tilde{T} := \{t \in T : t \text{ is dead in } z\}$ be the set of all transition which are dead in z. Remark that, due to definition 3.7, all places are dead or live in z^0 too. The following argumentatin will be repeated for $i \geq 0$.

Let $\tilde{T}_i := \left\{t \in T : t \text{ is dead in } z^i \wedge \exists p \in {}^\bullet t : p \text{ is dead in } z^i \wedge m^i(p_i) < V(p_i, t)\right\} \subseteq \tilde{T} \subseteq T$ be the set of all transitions which are dead in z^i and which have a pre-place, not enough marked for enabling the appropriate transition. If $\tilde{T} \setminus \tilde{T}_i \neq \emptyset$ then choose $t_i \in \tilde{T} \setminus \tilde{T}_i$ and fix it. Due to lemma 3.14 there is a state $z^{i+1} = (m^{i+1}, h^{i+1}) \in RS_{\mathcal{Z}}(z^i)$ such that

$$\exists p \in {}^\bullet t_i : p \text{ is dead in } z^{i+1} \wedge m^{i+1}(p) < V(p, t_i)$$

holds. Hence, $t_i \notin \tilde{T}_i s \subsetneq \tilde{T}_{i+1} \ni t_i$ is true.

Since the number of transitions is finite the sequence $(\tilde{T}_i)_{i \in \mathbb{N}}$ will be constant from T_n for a $n \in \mathbb{N}$. Due to construction $\tilde{T}_n = \tilde{T}$ holds. □

Lemma 3.16. Let $\mathcal{Z} = (T, P, F, V, m, I)$ be a TPN with a live skeleton. Then $\mathcal{Z}$ is deadlock-free.

Proof: Every p-marking m of a reachable state $z \in RS_{\mathcal{Z}}$ is also a reachable marking of the skeleton $S(\mathcal{Z})$. Assuming that there is a state $z = (m, h)$ and $\mathcal{Z}$ is dead in z concludes that no transition is enabled in m. This is a contradiction to the liveness of the skeleton.

Theorem 2. For each TPN with the properies (V1), (V2) and (V3) it holds: When $S(\mathcal{Z})$ is live than $\mathcal{Z}$ is live, as well.

Proof: Assume that the skeleton $S(\mathcal{Z})$ is live but the TPN $\mathcal{Z}$ is not live.

Due to lemma 3.8 there is a state $z' \in RS_{\mathcal{Z}}$ such that every transition and every place is either dead or live in that state. Because of lemma 3.15 there is a state $z'' = (m'', h'') \in RS_{\mathcal{Z}}(z')$ such that every transition $t \in T$ which is dead in z'' has a pre-place, dead in z'' and has not enough token for enabling t, i.e. $\exists p \in {}^\bullet t\ : m''(p) < V(p, t)$. Let

$$\tilde{T} := \left\{ t \in T : t \text{ is dead in } z'' \right\} \text{ and } \tilde{P} := {}^\bullet\tilde{T}$$

be. Since $\mathcal{Z}$ is deadlock-free it follows because of lemma 3.16 that $T \setminus \tilde{T} \neq \emptyset$. Due to the construction of $\tilde{T}$ it holds:

$$\forall t \in \tilde{T}\ \exists p \in {}^\bullet t \cap \tilde{P} : m''(p) < V(p, t).$$

Let $t^* \in \tilde{T}$ be an arbitrary transition. Since $S(\mathcal{Z})$ is live and m'' is a reachable marking in $S(\mathcal{Z})$ there is a transition sequence $\sigma = w_1 \ldots w_r$ such that $m'' \xrightarrow{\sigma} \xrightarrow{t^*}$ is feasible in the skeleton. Now, because a pre-place of t^* exists which is dead in z'' and has not enough tokens for enabling t^* there is a transition $w_i \in \sigma$ and $w_i \in \tilde{T}$. Let i be minimal with this property, i.e.

$$w_i \in \tilde{T} \wedge \forall 1 \leq j < i : w_j \notin \tilde{T}.$$

Due to the same reasone for $w_i \in \tilde{T}$ it holds:

$$\exists w_j \in w_1 \ldots w_{i-1} : w_j \in \tilde{T}.$$

This is a contradiction to the minimality of i and therefore $\mathcal{Z}$ has to be live. □

4. Conclusions

In this paper we manage to give an enlargement of the set of TPNs which are time-independent live, that means their liveness does not depend on the time. The proper enlargement refers to the property "generalized ES nets". For more than fifteen years the proof has been done for generalized EFC nets. However, a generalization of the "old" proof for generalized ES nets does not work.

It is important to know that the properties (V1), (V2) and (V3) cannot be done weaker (cf. [6]). Thus, for the future, we are looking for new time-independent live classes of TPNs: Especially we will consider BFC nets, defined in [3] and generalize them in the same manner as we have done with the EFC nets and the ES nets. However, such a class will be defined by dynamic properties, which are not decidable for TPNs in general.

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