On miniaturized problems in parameterized complexity theory

Yijia Chen and Jörg Flum

Abteilung für Mathematische Logik, Universität Freiburg, Eckerstr. 1,
79104 Freiburg, Germany.
chen@zermelo.mathematik.uni-freiburg.de Joerg.Flum@math.uni-freiburg.de

Abstract. We introduce a general notion of miniaturization of a problem that comprises the different miniaturizations of concrete problems considered so far. We develop parts of the basic theory of miniaturizations in this general framework. Using the appropriate logical formalism, we show that the miniaturization of a definable problem in W[t] lies in W[t], too. In particular, the miniaturization of the dominating set problem is in W[2].

1 Introduction

Parameterized complexity theory provides a framework for a refined complexity analysis of algorithmic problems that are intractable in general. Central to the theory is the notion of *fixed-parameter tractability*, which relaxes the classical notion of tractability, polynomial time computability, by admitting algorithms whose runtime is exponential, but only in terms of some *parameter* that is usually expected to be small. Let FPT denote the class of all fixed-parameter tractable problems. A well-known example of a problem in FPT is the vertex cover problem, the parameter being the size of the vertex cover we ask for.

As a complexity theoretic counterpart, a theory of *parameterized intractability* has been developed. In classical complexity, the notion of NP-completeness is central to a nice and simple theory for intractable problems. Unfortunately, the world of parameterized intractability is more complex: there is a big variety of seemingly different classes of parameterized intractability. For a long while, the smallest complexity class of parameterized intractable problems considered in the literature was W[1], the first class of the so-called W-hierarchy. (In particular, FPT $\subseteq$ W[1]; and FPT $\neq$ W[1] would imply PTIME $\neq$ NP.)

Recently, this situation has changed: In [4], Downey et al. consider various problems in W[1] that, apparently, are not W[1]-hard. Most of them are "miniaturizations" of well-studied problems in parameterized complexity theory; for example, mini-Circuit Sat is the problem that takes a circuit C of size $\leq k \cdot \log m$, where k is the parameter and m in unary is part of the input, and asks whether C is satisfiable. This problem is called a miniaturization of Circuit Sat, as the size ($\leq k \cdot \log m$) of C is small compared with m (under the basic assumption of parameterized complexity that the parameter k is small too). In [4], Downey et al. introduce the class MINI[1] as the class of parameterized problems fpt-reducible to mini-Circuit Sat. MINI[1] now provides very nice connections between classical complexity and parameterized complexity as it is known that FPT = MINI[1] if and only if n variable 3-Sat can be solved in time $2^{o(n)}$. This equivalence stated in [4] is based on a result of Cai and Juedes [1].

Besides this "miniaturization route", a second route to MINI[1] has been considered by Fellows in [7]; he calls it the "renormalization route" to MINI[1]. He "renormalizes" the parameterized VERTEX COVER problem and considers the so-called $k \cdot \log n$ VERTEX COVER problem: It takes as input a graph G and as parameter a natural number k; it asks if G has vertex cover of size $k \cdot \log n$. This problem turns out to be MINI[1]-complete (cf. [7]).

Before outlining the purpose and the contents of this paper let us give two quotations, the first one from Fellows' paper [7] and the second one from Downey's paper [3]:

> Dozens of renormalized FPT problems and miniaturized arbitrary problems are now known to be MINI[1]-complete. However, what is known is quite problem specific.
>
> Can the hierarchy [starting with MINI[1]] be extended?

Among others, in this paper we try to develop the theory of miniaturized problems on a more abstract level and we address the problems mentioned in these quotations. Concerning the second problem, even though we introduce a hierarchy of complexity classes, we conjecture, among others encouraged by the results of this paper, that the world of parameterized intractability in W[1] is so rich that, probably, there are various more or less natural hierarchies in W[1].

We sketch the content of the different sections. In Section 2 we give the necessary preliminaries. In particular, we introduce the notion of a size function, a polynomial time function $\| \ \|$ defined on the inputs x of a given problem with the property that the length $|x|$ of x is polynomially bounded in $\|x\|$. For example, for a graph $G = (V, E)$, natural choices could be $|V|$, the number of vertices, or $|V| + |E|$, the number of vertices and edges, or $\Theta(|V| + |E| \cdot \log |V|)$, the total length of its binary description; but, in general, $|E|$ is not a size function.

In Section 3, for a given size function $\| \ \|$, we define the concept of the miniaturization $\text{mini}^{\| \ \|}\text{-}Q$ of an arbitrary problem Q. Now, a proof essentially due to Cai and Juedes [1] goes through for this concept showing that $\text{mini}^{\| \ \|}\text{-}Q$ is fixed-parameter tractable just in case Qx is solvable in time $2^{o(\|x\|)}$. In Proposition 2 we extend the well-known fact that a linear reduction from Q to Q' yields an fpt-reduction from the miniaturization of Q to that of Q' and essentially show that the existence of a linear reduction from Q to Q' is *equivalent* to the existence of an fpt-reduction of the miniaturization of Q to that of Q' that is linear with respect to the parameters. Perhaps therefore, there are so many not fpt-equivalent miniaturizations.

There is a way of defining parameterized problems by means of first-order formulas with a free *set* variable X that has been dubbed *Fagin-definability* in [8], since it is related to Fagin's theorem characterizing NP as the class of Σ^1_1-definable problems. For example, the parameterized clique problem is Fagin-definable by the formula

$$\forall y \forall z ((Xy \wedge Xz \wedge y \neq z) \rightarrow Eyz).$$

In [5], Downey et al. showed that W[t], the tth class of the W-hierarchy, contains all parameterized problems Fagin-defined by Π_t-formulas and conversely, there are W[t]-complete problems Fagin-defined by Π_t-formulas. In general, the miniaturization may increase the computational complexity of a problem; e.g., the parameterized vertex cover problem is fixed-parameter tractable while its miniaturization is not (unless MINI[1] = FPT). But the miniaturization of every Fagin-definable problem in W[t] lies in W[t], too. We prove this result in Section 4 for $t = 1$ generalizing an idea used in [4] for the independent set problem. Some miniaturized problems considered in the literature can be regarded as miniaturization of *unweighted* Fagin-definable problems, a concept we introduce in this paper. In Section 5 we prove that in a certain sense weighted and unweighted definable problems have the same computational complexity. Using this result, we get the result on Fagin-definable problems for arbitrary t.

As mentioned above, Π_1-formulas of the form $\varphi(X) = \forall x_1 \dots \forall x_t \psi(X)$ with a set variable X and with quantifier-free $\psi(X)$ are used to obtain the Fagin-definable problems in W[1]. We obtain a hierarchy of classes within W[1] taking the length t of the block of quantifiers into consideration. We study the basic properties of this hierarchy; in particular, we show that the (appropriate) miniaturization of t-SAT is complete in the tth class of this hierarchy. Recall that Impagliazzo and Paturi [10] have shown that, assuming the exponential-time hypothesis, the complexity of t-SAT increases with t.

So far, when comparing the complexity of miniaturized and other parameterized problems, we used many-one reductions (more precisely, FPT many-one reductions). In some papers, Turing reductions have been considered. As we show in Section 7, most problems studied in this paper are Turing equivalent.

Finally, in Section 8 we deal with renormalizations. Besides the renormalization of the vertex cover problem introduced in Fellows [7], we consider a slightly different renormalization and also show its fpt-equivalence to the miniaturization. We shall see that this result cannot be extended to arbitrary Fagin-definable problems, in particular not to the clique problem.

Due to space limitations, we have to defer the proofs to the full version of the paper.

2 Preliminaries

We use standard terminology (cf. [6, 8]) and therefore, only briefly recall some notations, definitions, and results and introduce the concept of size function.

2.1 First-order logic and propositional logic

A *vocabulary* τ is a finite set of relation symbols. A *(relational) structure* $\mathcal{A}$ of vocabulary τ (or, simply structure), consists of a *finite* set A called the *universe*, and an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r-ary relation symbol $R \in \tau$.

For example, let $\tau_{\text{circ}} := \{E, I, O, G_\wedge, G_\vee, G_\neg\}$, where E is a binary relation symbol and I, O, $G_\wedge$, $G_\vee$, $G_\neg$ are unary relation symbols. We view Boolean circuits as τ_{circ}-structures

$$\mathcal{C} = (C, E^{\mathcal{C}}, I^{\mathcal{C}}, O^{\mathcal{C}}, G_\wedge^{\mathcal{C}}, G_\vee^{\mathcal{C}}, G_\neg^{\mathcal{C}}),$$

where $(C, E^{\mathcal{C}})$ is the directed acyclic graph underlying the circuit, $I^{\mathcal{C}}$ is the set of all input nodes, $O^{\mathcal{C}}$ just contains the output node, $G_\wedge^{\mathcal{C}}$, $G_\vee^{\mathcal{C}}$, and $G_\neg^{\mathcal{C}}$ are the sets of and-gates, or-gates (and-gates and or-gates of arbitrary arity), and negation-gates, respectively. The *weight* of a truth value assignment to the input nodes of $\mathcal{C}$ is the number of input nodes set to TRUE by the assignment.

Often for graphs we shall use the more common notation $G = (V, E)$ (or, $G = (V(G), E(G))$), where V is the set of vertices and E the set of edges.

We define the *size* $\|\mathcal{A}\|_0$ of a τ-structure $\mathcal{A}$ to be the number

$$\|\mathcal{A}\|_0 := |A| + \sum_{R \in \tau} \text{arity}(R) \cdot |R^{\mathcal{A}}| \cdot \log |A|.$$

In fact, the length of a reasonable encoding of $\mathcal{A}$ as a string is $\Theta(\|\mathcal{A}\|_0)$.

For $t \geq 1$, by Π_t we denote the class of all first-order formulas of the form

$$\forall x_{11} \ldots \forall x_{1k_1} \exists x_{21} \ldots \exists x_{2k_2} \ \ldots \ Q x_{t1} \ldots Q x_{tk_t} \ \psi,$$

where $Q = \forall$ if t is odd and $Q = \exists$ otherwise, and where ψ is quantifier-free. Σ_t-formulas are defined analogously starting with a block of existential quantifiers.

Formulas of propositional logic are built up from *propositional variables* $X_1, X_2, \ldots$ by taking conjunctions, disjunctions, and negations. We distinguish between *small conjunctions*, denoted by $\wedge$, which are just conjunctions of two formulas, and *big conjunctions*, denoted by $\bigwedge$, which are conjunctions of arbitrary finite sets of formulas. Analogously, we distinguish between *small disjunctions*, denoted by $\vee$, and *big disjunctions*, denoted by $\bigvee$.

For $t \geq 0$ and $d \geq 1$ we define the sets $\Gamma_{t,d}$ and $\Delta_{t,d}$ of propositional formulas by induction on t (here, by $(\lambda_1 \wedge \ldots \wedge \lambda_r)$ we mean the iterated small conjunction $((\ldots((\lambda_1 \wedge \lambda_2) \wedge \lambda_3) \ldots) \wedge \lambda_r)$:

$$\begin{aligned}
\Gamma_{0,d} &:= \{(\lambda_1 \wedge \ldots \wedge \lambda_r) \mid \lambda_1, \ldots, \lambda_r \text{ literals and } r \leq d\}, \\
\Delta_{0,d} &:= \{(\lambda_1 \vee \ldots \vee \lambda_r) \mid \lambda_1, \ldots, \lambda_r \text{ literals and } r \leq d\}, \\
\Gamma_{t+1,d} &:= \{\textstyle\bigwedge \Pi \mid \Pi \subseteq \Delta_{t,d}\}, \\
\Delta_{t+1,d} &:= \{\textstyle\bigvee \Pi \mid \Pi \subseteq \Gamma_{t,d}\}.
\end{aligned}$$

Let CNF denote the class of all propositional formulas in conjunctive normal form. A formula is in d-CNF, if it is a conjunction of disjunctions of at most d literals. Often, we identify a formula $\alpha \in d$-CNF with a τ_d-structure $\mathcal{A}(\alpha)$. Here $\tau_d := \{N, C\}$, where N is binary and C is d-ary, and $\mathcal{A}(\alpha)$ has the set $\{X, \neg X \mid X \text{ variable of } \alpha\}$ as universe and

$$\begin{aligned}
N^{\mathcal{A}(\alpha)} &:= \{(X, \neg X) \mid X \text{ variable of } \alpha\}; \\
C^{\mathcal{A}(\alpha)} &:= \{(\lambda_1, \ldots, \lambda_d) \mid (\lambda_1 \vee \ldots \vee \lambda_d) \text{ is a clause of } \alpha\},
\end{aligned}$$

w.l.o.g. we assume each clause has exactly d literals. A propositional formula α is *k-satisfiable*, if there is an assignment satisfying α of weight k (i.e., setting exactly k variables of α to TRUE).

2.2 Size functions

Let Σ be an alphabet. For a string $x \in \Sigma^*$, we denote its length by $|x|$.

Definition 1. *A function* $\| \; \| : \Sigma^* \to \mathbb{N}$ *is a* size function*, if it is computable in polynomial time and if, for some* $c \in \mathbb{N}$*,*

$$|x| \leq \|x\|^c$$

holds for all $x \in \Sigma^*$*.*

In particular, $| \; |$ is a size function. The function $\| \; \|_0$ (cf. Section 2.1) defined for τ-structures (more precisely, for the encodings of τ-structures by strings) is a size function. We introduce further size functions for τ-structures:

$$\|\mathcal{A}\|_+ := |A| + \sum_{R \in \tau} \text{arity}(R) \cdot |R^{\mathcal{A}}|,$$
$$\|\mathcal{A}\|_- := |A|.$$

Note that

- for a graph G with n vertices and m edges: $\|G\|_- = n$ and $\|G\|_+ = \Theta(n+m)$;
- for a circuit $\mathcal{C}$ with n gates and m lines: $\|\mathcal{C}\|_- = n$ and $\|\mathcal{C}\|_+ = \Theta(n+m)$;
- for a propositional formula $\alpha \in d$-CNF with n variables and m clauses: $\|\alpha\|_-$ $(:= \|\mathcal{A}(\alpha)\|_-)$ $= \Theta(n)$ and $\|\alpha\|_+$ $(:= \|\mathcal{A}(\alpha)\|_+)$ $= \Theta(n+m)$. [1]

2.3 Parameterized Complexity

A *parameterized problem* is a set $Q \subseteq \Sigma^* \times \Pi^*$, where Σ and Π are finite alphabets. If $(x, y) \in \Sigma^* \times \Pi^*$ is an instance of a parameterized problem, we refer to x as the *input* and to y as the *parameter*.

A parameterized problem $Q \subseteq \Sigma^* \times \Pi^*$ is *fixed-parameter tractable*, if there are a computable function $f : \mathbb{N} \to \mathbb{N}$, a polynomial p, and an algorithm that, given a pair $(x, y) \in \Sigma^* \times \Pi^*$, decides if $(x, y) \in Q$ in at most $f(|y|) \cdot p(|x|)$ steps.

FPT denotes the complexity class consisting of all fixed-parameter tractable parameterized problems.

Of course, we may view any parameterized problem $Q \subseteq \Sigma^* \times \Pi^*$ as a classical problem, say, in the alphabet obtained from $\Sigma \cup \Pi$ by adding new symbols '(', ',', ')'.

We mainly consider parameterized problems $Q \subseteq \Sigma^* \times \mathbb{N}$; so we give most definitions and state most results only for this case.

To illustrate our notation, let us give two examples of parameterized problems, the *weighted satisfiability problem* WSAT(Θ) for a class Θ of propositional formulas and the parameterized model-checking problem p-MC($\Sigma_1[\tau]$) for Σ_1-sentences of vocabulary τ:

WSAT(Θ)	
Input:	α in Θ.
Parameter:	$k \in \mathbb{N}$.
Problem:	Decide if α is k-satisfiable.

p-MC($\Sigma_1[\tau]$)	
Input:	A τ-structure $\mathcal{A}$.
Parameter:	$\varphi \in \Sigma_1[\tau]$.
Problem:	Decide if $\mathcal{A}$ satisfies φ.

[1] Note that for arbitrary propositional formulas the number of variables does not define a size function; for formulas α in d-CNF we obtain a size function, since we identify α with $\mathcal{A}(\alpha)$. Equivalently, we could restrict ourselves to formulas α that contain every clause at most once.

For computable functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \in o^{\text{eff}}(g)$, if $f \in o(g)$ holds in an effective way, i.e., if there is a computable function h such that, given any $\ell \in \mathbb{N}$ with $\ell > 0$, we have $f(m)/g(m) \leq 1/\ell$ for all $m \geq h(\ell)$.

Essentially the idea underlying the proof of Proposition 2 can be used to show that the effective versions of two notions of *subexponential time*, namely $\mathbf{DTIME}(2^{o(n)})$ and $\bigcap_{\epsilon>0} \mathbf{DTIME}(2^{\epsilon \cdot n})$ coincide:

Proposition 1. *For a classical problem $Q \subseteq \Sigma^*$ the following are equivalent.*

1. $Q \in \mathbf{DTIME}(2^{o^{\text{eff}}(n)})$*, i.e.,* $Q \in \mathbf{DTIME}(2^{o(g)})$ *for some function* $g \in o^{\text{eff}}(\text{id})$*, where* id *denotes the identity function on* $\mathbb{N}$*.*
2. *For every rational $\epsilon > 0$, there is an algorithm $\mathbb{A}_\epsilon$ deciding Q in time $O(2^{\epsilon \cdot n})$. Moreover, $\mathbb{A}_\epsilon$ can be computed from ϵ.*

Lemma 1 (Cai and Juedes [1]). *Let $Q \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem. Assume that there is an algorithm solving Qxk in time $2^{h(f(k) \cdot \log |x|)}$ for some computable function $h \in o^{\text{eff}}(\text{id})$. Then Qxk is solvable in time $g(k) + O(|x|)$ for some computable function g. In particular, $Q \in \text{FPT}$.*

Complementing the notion of fixed-parameter tractability, there is a theory of parameterized intractability. It is based on the notion of fpt-*reduction* (the natural notion of many-one reduction of parameterized complexity theory, see [6, 8]).

We write $Q \leq^{\text{fpt}} Q'$, if there is an fpt-reduction from Q to Q', and $Q =^{\text{fpt}} Q'$ if ($Q \leq^{\text{fpt}} Q'$ and $Q' \leq^{\text{fpt}} Q$). We set

$$[Q]^{\text{fpt}} := \{Q' \mid Q' \leq^{\text{fpt}} Q\} \quad \text{and} \quad [\mathrm{C}]^{\text{fpt}} := \bigcup_{Q \in \mathrm{C}} [Q]^{\text{fpt}}$$

for a class C of parameterized problems. For $t \geq 1$, the class W[t] is defined by

$$\mathrm{W}[t] := \big[\{\mathrm{WS\small{AT}}(\Gamma_{t,d}) \mid d \in \mathbb{N}\}\big]^{\text{fpt}}.$$

Clearly, FPT $\subseteq$ W[1] $\subseteq$ W[2] ... and it is conjectured that FPT $\neq$ W[1] (which would imply PTIME $\neq$ NP). It is well-known that

$$\mathrm{W}[1] = \big[\{p\text{-MC}(\Sigma_1[\tau]) \mid \tau \text{ a vocabulary}\}\big]^{\text{fpt}}.$$

3 The miniaturization of an arbitrary problem

In this section, for a classical problem Q and a size function $\| \ \|$, we introduce its miniaturization $\text{mini}^{\| \ \|}\text{-}Q$, a parameterized problem, and study the relationship between the complexity of Q and $\text{mini}^{\| \ \|}\text{-}Q$.

Definition 2. *Let $Q \subseteq \Sigma^*$ and let $\| \ \| : \Sigma^* \to \mathbb{N}$ be a size function. The* parameterized miniaturization $\text{mini}^{\| \ \|}\text{-}Q$ *with respect to $\| \ \|$ is the parameterized problem:*

$\text{mini}^{\| \ \|}\text{-}Q$
Input: $n, k \in \mathbb{N}$ in unary, and $x \in \Sigma^*$.
Parameter: k.
Problem: Decide if $\|x\| \leq k \cdot \log n$ and $x \in Q$.

Remark 1. a) Let $Q \subseteq \Sigma^*$ and $\| \ \|$ be a size function with $|x| \leq \|x\|^c$. The condition $\|x\| \leq k \cdot \log n$ can be checked in time polynomial in k and n only. Therefore, often the problem $\text{mini}^{\| \ \|}\text{-}Q$ is presented in the more appealing form:

$\text{mini}^{\| \ \|}\text{-}Q$
Input: $n, k \in \mathbb{N}$ in unary, $x \in \Sigma^*$ with $\|x\| \leq k \cdot \log n$.
Parameter: k.
Problem: Decide if $x \in Q$.

b) Arguing similarly as in part a), one shows that if $\text{mini}^{\| \ \|}\text{-}Q$ is in FPT, then there is an algorithm solving $\text{mini}^{\| \ \|}\text{-}Q$ (on instance n, k, x) in $\leq f(k) \cdot p(n)$ steps for some computable function f and some polynomial p.
c) If $Q \in \text{PTIME}$, then $\text{mini}^{\| \ \|}\text{-}Q \in \text{FPT}$.

The following result relates the fixed-parameter tractability of $\text{mini}^{\| \ \|}\text{-}Q$ with the solvability of Q in subexponential time. Its proof uses an idea of [1] in the form presented in [4].

Proposition 2. *For $Q \subseteq \Sigma^*$ and any size function $\| \ \| : \Sigma^* \to \mathbb{N}$ the following are equivalent:*

1. *Q is solvable in time $2^{o^{\text{eff}}(\|x\|)}$, i.e., in time $2^{h(\|x\|)}$ for some $h \in o^{\text{eff}}(\text{id})$.*
2. *$\text{mini}^{\| \ \|}\text{-}Q \in \text{FPT}$.*
3. *There is an algorithm that, for every instance n, k, x of $\text{mini}^{\| \ \|}\text{-}Q$ with $\|x\| \leq k \cdot \log n$ decides if $(n, k, x) \in \text{mini}^{\| \ \|}\text{-}Q$ in time $f(k) + O(n)$ for some computable f.*

The implication from (1) to (2) in the following proposition corresponds to the well-known fact that a polynomial time "linear size" reduction between two problems yields an fpt-reduction of their miniaturizations:

Proposition 3. *Let $Q_1 \subseteq \Sigma_1^*$ and $Q_2 \subseteq \Sigma_2^*$ and let $\| \ \|_i : \Sigma_i^* \to \mathbb{N}$ be a size function for $i = 1, 2$. Then, the following are equivalent:*

1. *There is a function $f : \Sigma_1^* \to \Sigma_2^*$ computable in time $2^{o^{\text{eff}}(\|x\|_1)}$ such that for all $x \in \Sigma_1^*$, $\|f(x)\|_2 \in O(\|x\|_1)$ and such that f is a reduction of Q_1 to Q_2, i.e.,*
$$x \in Q_1 \iff f(x) \in Q_2.$$
2. *There is an fpt-reduction R from $\text{mini}^{\| \ \|_1}\text{-}Q_1$ to $\text{mini}^{\| \ \|_2}\text{-}Q_2$ such that for any instance (n_1, k_1, x_1) of $\text{mini}^{\| \ \|_1}\text{-}Q_1$ with $\|x_1\|_1 \leq k_1 \cdot \log n_1$ we have $R(n_1, k_1, x_1) = (n_2, k_2, x_2)$ with $k_2 = O(k_1)$ and $\|x_2\|_2 \leq k_2 \cdot \log n_2$.*

Remark 2. a) Take as Q_1 a language in $\mathbf{DTIME}(2^{O(|x|)}) \setminus \mathbf{DTIME}(2^{o(|x|)})$ and as Q_2 a language in $\mathbf{DTIME}(2^{o^{\text{eff}}(|x|)})$ complete for EXPTIME under polynomial time reductions. In particular, there is a polynomial time reduction from Q_1 to Q_2. By Proposition 3, $\text{mini}^{| \ |}\text{-}Q_1 \notin \text{FPT}$ and $\text{mini}^{| \ |}\text{-}Q_2 \in \text{FPT}$. Hence, there is no fpt-reduction from $\text{mini}^{| \ |}\text{-}Q_1$ to $\text{mini}^{| \ |}\text{-}Q_2$. This example shows that the condition "$\|f(x)\|_2 \in O(\|x\|_1)$" in (1) of the preceding proposition cannot be weakened to "$\|f(x)\|_2 \le q(\|x\|_1)$ for some polynomial q".

b) For a natural number $d \ge 1$ replace the condition $k_2 \in O(k_1)$ in (2) of Proposition 3 by $k_2 \in O(k_1^d)$. Then, along the lines of the preceding proof, one can show that there is a reduction f from Q_1 to Q_2 according to (1) satisfying $\|f(x)\|_2 \in O(\|x\|_1^d)$

We close this section with some examples. Let CIRCSAT, SAT, and t-SAT denote the satisfiability problem for circuits, for propositional formulas in CNF, and for propositional formulas in t-CNF, respectively. In Section 2.1, for a circuit $\mathcal{C}$, we defined $\|\mathcal{C}\|_0$, $\|\mathcal{C}\|_+$, and $\|\mathcal{C}\|_-$. Essentially they are the (total) size of an encoding of $\mathcal{C}$, the number of gates + the number of lines of $\mathcal{C}$, and the number of gates of $\mathcal{C}$, respectively. In the following, we abbreviate $\text{mini}^{\| \|_+}$-CIRCSAT and $\text{mini}^{\| \|_-}$-CIRCSAT by mini^+-CIRCSAT and mini^--CIRCSAT, respectively. The same notations are used for other problems. Taking as Q in Proposition 2 the problem CIRCSAT, we get the following result (cf. [1, 4]); it shows, for example, that mini^+-CIRCSAT $\in$ FPT is quite unlike.

Proposition 4. *1. For* $\| \| \in \{\| \|_+, \| \|_-\}$*:* $\text{mini}^{\| \|}$-CIRCSAT $\in$ FPT *if and only if there is a subexponential algorithm for* CIRCSAT*, i.e., if there is an algorithm with running time* $2^{o^{\text{eff}}(\|\mathcal{C}\|)}$ *checking if the circuit* $\mathcal{C}$ *is satisfiable.*
2. $\text{mini}^{\| \|_0}$-CIRCSAT $\in$ FPT*.*

By Proposition 3, the well-known linear reductions between CIRCSAT, SAT, and 3-SAT yield:

– mini^+-CIRCSAT $=^{\text{fpt}}$ mini^+-SAT $=^{\text{fpt}}$ mini^+-3-SAT.

Denote by VC, IS, and CLIQUE the vertex cover problem, the independent set problem, and the clique problem, respectively; e.g., the instances to CLIQUE consist of pairs (G, r), where $G = (V, E)$ is a graph and r is a natural number with $r \le |V|$. $(G, r) \in$ CLIQUE if and only if there is a clique of size r in G. We let $\|(G, r)\|_- = \|G\|_-$ and $\|(G, r)\|_+ = \|G\|_+$ and use the analogous notations for VC and IS. Clearly, the mini^+ versions of these problems are fpt-reducible to their mini^- versions. Using this fact and well-known linear reductions between the corresponding problems, we get the following fpt-reductions between their miniaturized versions:

– mini^+-3-SAT $\le^{\text{fpt}}$ mini^+ -VC $=^{\text{fpt}}$ mini^+-IS $\le^{\text{fpt}}$ mini^--IS $=^{\text{fpt}}$ mini^--VC $=^{\text{fpt}}$ mini^--CLIQUE.

4 The miniaturization of Fagin-definable problems

It is well-known that the parameterized halting problem HP for Turing machines, parameterized by the number of steps, is in W[1], even it is W[1]-complete. As pointed out in Downey [3], it is unknown whether $\text{mini}^{| |}$-HP $\in$ W[1]; we conjecture that this is not the case. In this section we show that the miniaturizations of Fagin-definable problems in W[1] are themselves in W[1].

We start by recalling the definition of Fagin-definable problem. Let τ be a vocabulary and C a class of τ-structures decidable in polynomial time. Let $\varphi(X)$ be a first-order formula of vocabulary τ that contains a free set variable X (that is, X is a unary second-order variable); it defines a problem $W_{\varphi(X)}$ on C:

$W_{\varphi(X)}(\mathrm{C})$ *Input:* A τ-structure $\mathcal{A}$ in C.   *Parameter:* $r \in \mathbb{N}$ with $r \leq \lvert A \rvert$.   *Problem:* Is there a subset S of A of cardinality r satisfying $\varphi(X)$ in $\mathcal{A}$, i.e., with $\mathcal{A} \models \varphi(S)$?

We say that $\varphi(X)$ *Fagin-defines* $W_{\varphi(X)}(\mathrm{C})$ *on* C and that a parameterized problem $Q \subseteq \mathrm{C}$ is *Fagin-definable*, if $Q = W_{\varphi(X)}(\mathrm{C})$ for some $\varphi(X)$.

For example, the parameterized vertex cover problem, the independent set problem, and the dominating set problem are Fagin-defined on the class GRAPH of all graphs by $\forall y \forall z \big(Eyz \to (Xy \vee Xz)\big)$, by $\forall y \forall z \big((Xy \wedge Xz) \to \neg Eyz\big)$, and by $\forall y \exists z \big(Xz \wedge (y = z \vee Eyz)\big)$, respectively.

If C is the class of all τ-structures, we denote $W_{\varphi(X)}(\mathrm{C})$ by $W_{\varphi(X)}$. For notational simplicity, we formulate most results for Fagin-definable problems $W_{\varphi(X)}$. Their extensions to Fagin-definable problems $W_{\varphi(X)}(\mathrm{C})$ are easy.

So far, we defined the miniaturization for classical problems only. Here and later, when speaking of the miniaturization of a parameterized problem Q, we consider Q as a classical problem as explained in Section 2.3. Again for a structure $\mathcal{A}$ and $r \in \mathbb{N}$ with $r \leq \lvert A \rvert$, we set $\lVert (\mathcal{A}, r) \rVert_- = \lVert \mathcal{A} \rVert_-$ and $\lVert (\mathcal{A}, r) \rVert_+ = \lVert \mathcal{A} \rVert_+$. For example for a formula $\varphi(X)$ of vocabulary τ, mini$^-$-$W_{\varphi(X)}$ is the problem

mini$^-$-$W_{\varphi(X)}$ *Input:* $n, k \in \mathbb{N}$ in unary, a τ-structure $\mathcal{A}$ with $\lvert A \rvert \leq k \cdot \log n$, and $r \in \mathbb{N}$ with $r \leq \lvert A \rvert$.   *Parameter:* k.   *Problem:* Is there a subset S of A of cardinality r with $\mathcal{A} \models \varphi(S)$?

It was shown in [5] (see [8]) that $\mathrm{W}[t] = \big[\{W_{\varphi(X)} \mid \varphi(X) \text{ a } \Pi_t\text{-formula}\}\big]^{\mathrm{fpt}}$.

The following theorem shows that the miniaturization with respect to $\lVert\ \rVert_-$ (and hence, its miniaturization with respect to $\lVert\ \rVert_+$) of a Fagin-definable problem in W[1], lies in W[1], too. In [4], it was shown that mini$^-$-IS is reducible to the parameterized independent set problem, a problem in W[1]. There, a Turing reduction was exhibited. We generalize the idea underlying this reduction appropriately to obtain our result.

Theorem 1. *Let $\varphi(X)$ be a Π_1-formula. Then,* mini$^-$-$W_{\varphi(X)} \in \mathrm{W}[1]$.

Above we gave definitions of VC and IS by means of Π_1-formulas, hence:

Corollary 1. mini$^-$-VC*,* mini$^-$-IS $\in \mathrm{W}[1]$.

5 The miniaturization of unweighted definable problems

In general, in parameterized complexity we consider *weighted* satisfiability problems; often, in the context of miniaturizations they are considered in the *unweighted* form. In this section, for Fagin-definable problems we show that the miniaturized weighted and unweighted problems have the same computational complexity. Using this result, we prove, for every $t \geq 1$, that the miniaturization of a Fagin-definable problem in W[t] lies in W[t], too.

Definition 3. *Let τ be a vocabulary. Let $\varphi(X)$ be a first-order formula of vocabulary τ with the free set variable X; it defines a classical problem $U_{\varphi(X)}$ given by:*

$U_{\varphi(X)}$
Input: A τ-structure $\mathcal{A}$.
Problem: Is there a subset S of A with $\mathcal{A} \models \varphi(S)$?

Theorem 2. *For $q \geq 3$ and $t \geq 1$:*

1. $\left[\{\text{mini}^{+}\text{-}U_{\varphi(X)} \mid \varphi(X) \in \Pi_{1,q}\}\right]^{\text{fpt}} = \left[\{\text{mini}^{+}\text{-}W_{\varphi(X)} \mid \varphi(X) \in \Pi_{1,q}\}\right]^{\text{fpt}}$.
2. $\left[\{\text{mini}^{-}\text{-}U_{\varphi(X)} \mid \varphi(X) \in \Pi_{1,q}\}\right]^{\text{fpt}} = \left[\{\text{mini}^{-}\text{-}W_{\varphi(X)} \mid \varphi(X) \in \Pi_{1,q}\}\right]^{\text{fpt}}$.
3. $\left[\{\text{mini}^{+}\text{-}U_{\varphi(X)} \mid \varphi(X) \in \Pi_{t}\}\right]^{\text{fpt}} = \left[\{\text{mini}^{+}\text{-}W_{\varphi(X)} \mid \varphi(X) \in \Pi_{t}\}\right]^{\text{fpt}}$.
4. $\left[\{\text{mini}^{-}\text{-}U_{\varphi(X)} \mid \varphi(X) \in \Pi_{t}\}\right]^{\text{fpt}} = \left[\{\text{mini}^{-}\text{-}W_{\varphi(X)} \mid \varphi(X) \in \Pi_{t}\}\right]^{\text{fpt}}$.

By Theorem 1 this implies:

Corollary 2. *If $\varphi(X) \in \Pi_1$ then $\text{mini}^{-}\text{-}U_{\varphi(X)}, \text{mini}^{+}\text{-}U_{\varphi(X)} \in \text{W}[1]$.*

Theorem 3. *Let $t \geq 1$ and $\varphi(X) \in \Pi_t$. Then $\text{mini}^{+}\text{-}W_{\varphi(X)}$, $\text{mini}^{-}\text{-}W_{\varphi(X)} \in \text{W}[t]$.*

Corollary 3. mini^{-}-Dominating Set $\in$ W[2].

Recall that the parameterized dominating set problem (p-DS) is complete for W[2], so Corollary 3 implies that if p-DS $\in$ FPT, then the dominating set problem can be decided in time $2^{o^{\text{eff}}(|V|)}$. Moreover by a refined argument, similar to that needed to prove Theorem 3, one can show the following result analogous to a result concerning the clique problem proved in [2].

Theorem 4. *If p-DS can be decided in time $f(k) \cdot n^{o^{\text{eff}}(k)}$ for a computable function f, then both the dominating set problem and the clique problem are solvable in time $2^{o^{\text{eff}}(|V|)}$.*

This improves a result stated in [7].

6 The M^{-}-hierarchy

The previous analysis suggests the definition of a hierarchy within W[1], taking the number of universal quantifiers into account: For $t \geq 1$, we introduce the class M$^{-}[t]$ by

$$\text{M}^{-}[t] := \left[\{\text{mini}^{-}\text{-}U_{\varphi(X)} \mid \varphi(X) \in \Pi_{1,t}\}\right]^{\text{fpt}}.$$

By Corollary 2, $\text{M}^{-}[t] \subseteq \text{W}[1]$; therefore,

$$\text{M}^{-}[1] \subseteq \text{M}^{-}[2] \subseteq \ldots \subseteq \text{M}^{-}[t] \subseteq \ldots \text{W}[1]. \tag{1}$$

By Theorem 2, for $t \geq 3$,

$$\text{M}^{-}[t] = \left[\{\text{mini}^{-}\text{-}W_{\varphi(X)} \mid \varphi(X) \in \Pi_{1,t}\}\right]^{\text{fpt}}.$$

Theorem 5. *For all $t \geq 2$,* mini^--t-SAT *is complete in* $\mathrm{M}^-[t]$*.*

It is well-known that 2-SAT is in PTIME, thus mini^--2-SAT $\in$ FPT by part c) of Remark 1. Hence, $(\mathrm{M}^-[1] =) \mathrm{M}^-[2] = \mathrm{FPT}$ by the preceding theorem.

Is the hierarchy in (1) (starting with $t = 2$) strict? By the preceding theorem, we know that mini^--t-SAT is $\mathrm{M}^-[t]$-complete; hence, in connection with this problem one should mention the result of Impagliazzo and Paturi [10] that, assuming the exponential-time hypothesis, the complexity of t-SAT increases with t.

7 MINI[1] and Turing reductions

In [4], the class MINI[1] (sometimes denoted by M[1]) was introduced:

$$\mathrm{MINI}[1] := [\text{mini}^+\text{-CIRCSAT}]^{\mathrm{fpt}}.$$

Since mini^+-CIRCSAT $=^{\mathrm{fpt}}$ mini^+-3-SAT $\leq^{\mathrm{fpt}}$ mini^--3-SAT, we know by Theorem 5 that $\mathrm{MINI}[1] \subseteq \mathrm{M}^-[3]$.

Sometimes (e.g., in [4, 7]), in connection with the class MINI[1], Turing reductions (more precisely, parameterized Turing reductions) have been considered; the fpt-reductions considered so far in this paper were many-one reductions.

From the point of view of Turing reductions nearly all problems considered here have the same complexity. In fact, it has been implicitly shown in Impagliazzo, Paturi and Zane [11] that for $t \geq 3$, there is a parameterized Turing reduction from mini^--t-SAT to mini^+-t-SAT, hence, these two problems are Turing equivalent. In particular, if we denote by $\mathrm{MINI}_T[1]$ and $\mathrm{M}_T^-[t]$ the closure under parameterized Turing reductions of MINI[1] and $\mathrm{M}^-[t]$, respectively, we have $\mathrm{MINI}_T[1] = \mathrm{M}_T^-[3]$, but also $\mathrm{M}_T^-[t] = \mathrm{M}_T^-[3]$ for all $t \geq 3$.

8 Renormalizations

In the context of miniaturized problems two "renormalizations" of VC have been considered, $(k \cdot \log n)^+$-VC and $(k \cdot \log n)^-$-VC. Let $\| \ \|$ be an arbitrary size function on the class of graphs. Define $(k \cdot \log n)^{\| \ \|}$-VC by

$(k \cdot \log n)^{\| \ \|}$-VC
Input: $G = (V, E)$.
Parameter: $k \in \mathbb{N}$.
Problem: Does G have a vertex cover of size $k \cdot \log \|G\|$?

Clearly, $(k \cdot \log n)^-$-VC and $(k \cdot \log n)^+$-VC denote $(k \cdot \log n)^{\| \ \|^-}$-VC and $(k \cdot \log n)^{\| \ \|^+}$-VC, respectively. We show that both problems, $(k \cdot \log n)^-$-VC and $(k \cdot \log n)^+$-VC, are fpt-equivalent to mini^--VC. The equivalence of the first and the third problem is stated in [7]. There, no proof is given and hence, we do not know if [7] refers to Turing reductions or to many-one reductions.

Theorem 6. $(k \cdot \log n)^-$-VC*,* $(k \cdot \log n)^+$-VC*, and* mini^--VC *are* fpt*-equivalent.*

Remark 3. In the same way, one could define the $k \cdot \log n$ renormalizations of various parameterized problems, for example, of all Fagin-definable ones. Since we have no substantial results for the general case, we introduced the notion of renormalization for the vertex cover problem directly. In fact, the preceding theorem does not generalize to CLIQUE:

Claim: If FPT $\neq$ W[1], then $(k \cdot \log n)^-$-CLIQUE $\not\leq^{\text{fpt}}$ mini$^-$-CLIQUE.

For the reader familiar with the terminology of [9], we state the following generalization of this claim:

> For $t \geq 2$, if W[t] $\neq$ FPT, then $(k \cdot \log n)^- W_{\varphi(X)} \not\leq^{\text{fpt}}$ mini$^- W_{\varphi(X)}$ for every generic $\Pi_{t/1}$-formula $\varphi(X)$.

In particular, for the dominating set problem, we have

If W[2] $\neq$ FPT, then $(k \cdot \log n)^-$-DS $\not\leq^{\text{fpt}}$ mini$^-$-DS.

9 Conclusions

We have introduced a general notion of miniaturization of a problem that comprises the different miniaturizations of concrete problems considered so far. Using the appropriate logical formalism, we were able to show that the miniaturizations of definable problems in W[t] are in W[t], too. Based on this logical formalism we introduced a hierarchy of complexity classes in W[1].

References

1. L. Cai and D. Juedes. Subexponential parameterized algorithms collapse the W-hierarchy. In *Proceedings of ICALP'01, LNCS 2076*, 2001.
2. J. Chen, X. Huang, I. Kanj and G. Xia. Linear FPT Reductions and Computational Lower Bounds. In *Proceedings of STOC'04*, 2004.
3. R. Downey. Parameterized complexity for the sceptic. In *Proceedings of the 18th IEEE Conference on Computational Complexity*, pages 147–168, 2003.
4. R. Downey, V. Estivill, M. Fellows, E. Prieto, and F. Rosamond. Cutting up is hard to do: the parameterized complexity of k-cut and related problems. In *Proceedings of CATS'03, ENTCS*, 78(0), 2003.
5. R.G. Downey, M.R. Fellows, and K. Regan. Descriptive complexity and the W-hierarchy. In P. Beame and S. Buss, editors, *Proof Complexity and Feasible Arithmetic*, volume 39 of *AMS-DIMACS Volume Series*, pages 119–134. AMS, 1998.
6. R.G. Downey and M.R. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.
7. M. Fellows. *New directions and new challenges in algorithm design and complexity, parameterized.*. In *Proceedings of WADS 2003, LNCS 2748*, 2003.
8. J. Flum and M. Grohe. Fixed-parameter tractability, definability, and model checking. *SIAM Journal on Computing*, 31(1):113–145, 2001.
9. J. Flum, M. Grohe, and M. Weyer. Bounded fixed-parameter tractability and $\log^2 n$ nondeterministic bits . In *Proceedings of ICALP'04*, 2004.
10. R. Impagliazzo and R. Paturi. Complexity of k-SAT. *JCSS*, 62(2): 367-375, 2001.
11. R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *JCSS*, 63(4):512-530, 2001.