The succinctness of first-order logic on linear orders (Full Version)

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Abstract

Succinctness is a natural measure for comparing the strength of different logics. Intuitively, a logic L_1 is more succinct than another logic L_2 if all properties that can be expressed in L_2 can be expressed in L_1 by formulas of (approximately) the same size, but some properties can be expressed in L_1 by (significantly) smaller formulas.

We study the succinctness of logics on linear orders that have the same expressive power as first-order logic. Our first theorem is concerned with the finite variable fragments of first-order logic. We prove that:

(i) Up to a polynomial factor, the 2- and the 3-variable fragments of first-order logic on linear orders have the same succinctness.

(ii) The 4-variable fragment is exponentially more succinct than the 3-variable fragment.

Our second main result compares the succinctness of first-order logic on linear orders with that of monadic second-order logic. We prove that the fragment of monadic second-order logic that has the same expressiveness as first-order logic on linear orders is non-elementarily more succinct than first-order logic.

1. Introduction

It is one of the fundamental themes of logic in computer science to study and compare the *strength* of various logics. Maybe the most natural measure of strength is the *expressive power* of a logic. By now, researchers from finite model theory, but also from more application driven areas such as database theory and automated verification, have developed a rich toolkit that has led to a good understanding of the expressive power of the fundamental logics (e.g. [3, 10]). It should also be said that there are clear limits to the understanding of expressive power, which are often linked to open problems in complexity theory.

In several interesting situations, however, one encounters different logics of the same expressive power. As an example, let us consider node selecting query languages for XML-documents. Here the natural deductive query language monadic datalog [7] and various automata based query "languages" [12, 13, 6] have the same expressive power as monadic second-order logic. XML-documents are usually modeled by labeled trees. Logics on trees and and strings also play an important role in automated verification. Of the logics studied in the context of verification, the modal μ-calculus is another logic that has the same expressive power as monadic second-order logic on ranked trees and strings, and linear time temporal logic LTL has the same expressive power as first-order logic on strings [11].

Succinctness is a natural measure for comparing the strength of logics that have the same expressive power. Intuitively, a logic L_1 is more succinct than another logic L_2 if all properties that can be expressed in L_2 can be expressed in L_1 by formulas of (approximately) the same size, but some properties can be expressed in L_1 by (significantly) smaller formulas.

For both expressiveness and succinctness there is a trade-off between the strength of a logic and the complexity of evaluating formulas of the logic. The difference lies in the way the complexity is measured. Expressiveness is related to *data complexity*, which only takes into account the size of the structure in which the formula has to be evaluated, whereas succinctness is related to the *combined complexity*, which takes into account both the size of the formula and structure [15].

Succinctness has received surprisingly little attention so far; a few scattered results are [16, 2, 1, 4]. In [8], we started a more systematic investigation. Specifically, we studied the succinctness of various logics on trees that all have the same expressive power as monadic second-order logic. While we

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were able to gain a reasonable picture of the succinctness of these logics, it also became clear that we are far from a thorough understanding of succinctness. In particular, very few techniques for proving lower bounds are available.

Indeed, most of the lower bound proofs use automata theoretic arguments, often combined with a clever encoding of large natural numbers that goes back to Stockmeyer [14]. In [8], these techniques were also combined with complexity theoretic reductions to prove lower bounds on succinctness under certain complexity theoretic assumptions. Wilke [16] used refined automata theoretic arguments to prove that CTL^+ is exponentially more succinct than CTL. Adler and Immerman [1] were able to improve Wilke's lower bound slightly, but what is more interesting is that they introduced games for establishing lower bounds on succinctness. These games vaguely resemble Ehrenfeucht-Fraïssé games, which are probably the most important tools for establishing inexpressibility results.

In this paper, we study the succinctness of logics on linear orders (without any additional structure) that have the same expressive power as first-order logic. In particular, we consider finite variable fragments of first-order logic. It is known and easy to see that even the 2-variable fragment has the same expressive power as full first-order logic on linear orders (with respect to Boolean and unary queries). We prove the following theorem:

Theorem 1.1. *(i) Up to a polynomial factor, the 2 and the 3-variable fragments of first-order logic on linear orders have the same succinctness.*

(ii) The 4-variable fragment of first-order logic on linear orders is exponentially more succinct than the 3-variable fragment. □

For the sake of completeness, let us also mention that full first-order logic is at most exponentially more succinct than the 3-variable fragment. It remains an open problem if an exponential gap between full first-order logic and the 4-variable fragment is needed, too.

Of course the main result here is the exponential gap in Theorem 1.1 (ii), but it should be noted that (i) is also by no means obvious. The theorem may seem very technical and not very impressive at first sight, but we believe that to gain a deeper understanding of the issue of succinctness it is of fundamental importance to master basic problems such as those we consider here first (similar, maybe, to basic inexpressibility results such as the inability of first-order logic to express that a linear order has even length). The main technical result behind both parts of the theorem is that a 3-variable first-order formula stating that a linear order has length m must have size at least $\frac{1}{2}\sqrt{m}$. Our technique for proving this result originated in the Adler-Immerman games, even though later it turned out that the proofs are clearer if the reference to the game is dropped.

There is another reason the gap in succinctness between the 3- and 4-variable fragments is interesting: It is a long standing open problem in finite model theory if, for $k \geq 3$, the k variable fragment of first-order logic is strictly less expressive than the $(k{+}1)$-variable fragment on the class of all ordered finite structures. This question is still open for all $k \geq 3$. Our result (ii) at least shows that there are properties that require exponentially larger 3-variable than 4-variable formulas.

Succinctness as a measure for comparing the strength of logics is not restricted to logics of the same expressive power. Even if a logic L_1 is more expressive than a logic L_2, it is interesting to know whether those properties that can be expressed in both L_1 and L_2 can be expressed more succinctly in one of the logics. Sometimes, this may even be more important than the fact that some esoteric property is expressible in L_1, but not L_2. We compare first-order logic with the more expressive monadic second-order logic and prove:

Theorem 1.2. *The fragment of monadic second-order logic that has the same expressiveness as first-order logic on linear orders is non-elementarily more succinct than first-order logic.* □

The paper is organized as follows: After the Preliminaries, in Section 3 we prove the main technical result behind Theorem 1.1. In Sections 4 and 5 we formally state and prove the two parts of the theorem. Finally, Section 6 is devoted to Theorem 1.2.

The present paper is the full version of the conference contribution [9].

2. Preliminaries

We write $\mathbb{N}$ for the set of non-negative integers.

We assume that the reader is familiar with first-order logic FO (cf., e.g., the textbooks [3, 10]). For a natural number k we write FO^k to denote the *k-variable fragment* of FO. The three variables available in FO^3 will always be denoted x, y, and z. We write $\text{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$ (resp., $\text{FO}^3(<)$) to denote the class of all FO^3-formulas of signature $\{<, \mathit{Succ}, \mathit{min}, \mathit{max}\}$ (resp., of signature $\{<\}$), with binary relation symbols $<$ and *Succ* and constant symbols *min* and *max*. In the present paper, such formulas will always be interpreted in finite structures where $<$ is a linear ordering, *Succ* the successor relation associated with $<$, and *min* and *max* the minimum and maximum elements w.r.t. $<$.

For every $N \in \mathbb{N}$ let $\mathcal{A}_N$ be the $\{<, \mathit{Succ}, \mathit{min}, \mathit{max}\}$-structure with universe $\{0, \ldots, N\}$, $<$ the natural linear ordering, $\mathit{min}^{\mathcal{A}_N} = 0$, $\mathit{max}^{\mathcal{A}_N} = N$, and *Succ* the relation with $(a, b) \in \mathit{Succ}$ iff $a{+}1 = b$. We identify the *class of linear orders* with the set $\{\mathcal{A}_N : N \in \mathbb{N}\}$.

For a structure $\mathcal{A}$ we write $\mathcal{U}^{\mathcal{A}}$ to denote $\mathcal{A}$'s universe. When considering FO^3, an *interpretation* is a tuple $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is one of the structures $\mathcal{A}_N$ (for some $N \in \mathbb{N}$) and $\alpha : \{x, y, z\} \to \mathcal{U}^{\mathcal{A}}$ (i.e., α is an *assignment* in $\mathcal{A}$). For a variable $v \in \{x, y, z\}$ and an element $a \in \mathcal{U}^{\mathcal{A}}$ we write $\alpha[\frac{a}{v}]$ to denote the assignment that maps v to a and that coincides with α on all other variables. If A is a set of interpretations and φ is an $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-formula, we write $A \models \varphi$ to indicate that φ is satisfied by *every* interpretation in A.

In a natural way, we view formulas as finite trees (precisely, as their *syntax trees*), where leaves correspond to the atoms of the formulas, and inner vertices correspond to Boolean connectives or quantifiers. We define the *size* $||\varphi||$ of φ to be the number of vertices of φ's syntax tree.

Definition 2.1. [Succinctness]
Let L_1 and L_2 be logics, let F be a class of functions from $\mathbb{N}$ to $\mathbb{N}$, and let $\mathcal{C}$ be a class of structures. We say that *L_1 is F-succinct in L_2 on $\mathcal{C}$* iff there is a function $f \in F$ such that for every L_1-sentence φ_1 there is an L_2-sentence φ_2 of size $||\varphi_2|| \leqslant f(||\varphi_1||)$ which is equivalent to φ_1 on all structures in $\mathcal{C}$. □

Intuitively, a logic L_1 being F-succinct in a logic L_2 means that F gives an upper bound on the size of L_2-formulas needed to express *all* of L_1. This definition may seem slightly at odds with the common use of the term "succinctness" in statements such as "L_1 is exponentially *more succinct* than L_2" meaning that there is *some* L_1-formula that is not equivalent to any L_2-formula of sub-exponential size. In our terminology we would rephrase this last statement as "L_1 is *not* $2^{o(m)}$-succinct in L_2" (here we interpret sub-exponential as $2^{o(m)}$, but of course this is not the issue). The reason for defining F-succinctness the way we did is that it makes the formal statements of our results much more convenient. We will continue to use statements such as "L_1 is exponentially more succinct than L_2" in informal discussions.

Example 2.2. $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$ is $\mathcal{O}(m)$-succinct in $\mathrm{FO}^3(<)$ on the class of linear orders, because $\mathit{Succ}(x, y)$ (resp., $x{=}\mathit{min}$, resp., $x{=}\mathit{max}$) can be expressed by the formula $(x{<}y) \wedge \neg\exists z \big((x{<}z) \wedge (z{<}y)\big)$ (resp., $\neg\exists y(y{<}x)$, resp., $\neg\exists y(x{<}y)$). □

3. Lower bound for FO^3

3.1. Lower Bound Theorem.

Before stating our main lower bound theorem, we need some more notation.

If S is a set we write $\mathcal{P}_2(S)$ for the set of all 2-element subsets of S. For a finite subset S of $\mathbb{N}$ we write $\mathit{MAX}\ S$ (respectively, $\mathit{MIN}\ S$) to denote the maximum (respectively, minimum) element in S. For integers m, n we define $\mathit{diff}(m, n) := m - n$ to be the difference between m and n. We define $<\text{-}\mathit{type}(m, n) \in \{<, =, >\}$ as follows:

$$
\begin{array}{llll}
\text{if} & m < n & \text{then} & <\text{-}\mathit{type}(m,n) := \text{“}<\text{”},\\
\text{if} & m = n & \text{then} & <\text{-}\mathit{type}(m,n) := \text{“}=\text{”},\\
\text{if} & m > n & \text{then} & <\text{-}\mathit{type}(m,n) := \text{“}>\text{”}.
\end{array}
$$

We next fix the notion of a *separator*. Basically, if A and B are sets of interpretations and δ is a separator for $\langle A, B\rangle$, then δ contains information that allows to distinguish every interpretation $\mathcal{I} \in A$ from every interpretation $\mathcal{J} \in B$.

Definition 3.1. [separator]
Let A and B be sets of interpretations.
A *potential separator* is a mapping

$$\delta : \mathcal{P}_2\big(\{\mathit{min}, \mathit{max}, x, y, z\}\big) \longrightarrow \mathbb{N}\,.$$

δ is called *separator for* $\langle A, B\rangle$, if the following is satisfied: For every $\mathcal{I} := (\mathcal{A}, \alpha) \in A$ and $\mathcal{J} := (\mathcal{B}, \beta) \in B$ there are $u, u' \in \{\mathit{min}, \mathit{max}, x, y, z\}$ with $u \neq u'$, such that $\delta\big(\{u, u'\}\big) \geqslant 1$ and

1. $<\text{-}\mathit{type}\big(\alpha(u), \alpha(u')\big) \neq <\text{-}\mathit{type}\big(\beta(u), \beta(u')\big)$ or
2. $\delta\big(\{u, u'\}\big) \geqslant \mathit{MIN}\,\big\{|\mathit{diff}\big(\alpha(u), \alpha(u')\big)|\,,\ |\mathit{diff}\big(\beta(u), \beta(u')\big)|\big\}$ and $\mathit{diff}\big(\alpha(u), \alpha(u')\big) \neq \mathit{diff}\big(\beta(u), \beta(u')\big)$. □

Note that δ is a separator for $\langle A, B\rangle$ if, and only if, δ is a separator for $\langle\{\mathcal{I}\}, \{\mathcal{J}\}\rangle$, for all $\mathcal{I} \in A$ and $\mathcal{J} \in B$. For simplicity, we will often write $\langle\mathcal{I}, \mathcal{J}\rangle$ instead of $\langle\{\mathcal{I}\}, \{\mathcal{J}\}\rangle$.

Let us now state a lemma on the existence of separators.

Lemma 3.2. *If A and B are sets of interpretations for which there exists an* $\mathrm{FO}^3(<)$*-formula ψ such that $A \models \psi$ and $B \models \neg\psi$, then there exists a* separator *δ for $\langle A, B\rangle$.* □

The proof is given in Appendix A.

Definition 3.3. [weight of δ]
Let δ be a potential separator for $\langle A, B\rangle$. We define the

(a) *border-distance* $b(\delta) :=$
$\mathit{MAX}\,\big\{\delta(\{\mathit{min},\ \mathit{max}\}),\ \delta(\{\mathit{min}, u\}) + \delta(\{u', \mathit{max}\}) :\ u, u' \in \{x, y, z\}\big\}$

(b) *center-distance* $c(\delta) :=$
$\mathit{MAX}\,\big\{\delta(p) + \delta(q) : p, q \in \mathcal{P}_2(\{x, y, z\}),\ p \neq q\big\}$

(c) *weight* $w(\delta) := \sqrt{c(\delta)^2 + b(\delta)}$. □

There is not much intuition we can give for this particular choice of weight function, except for the fact that it seems to be exactly what is needed for the proof of our main lower bound theorem. At least it will later, in Remark 3.12, become clear why the $\sqrt{\ }$-function is used for defining the weight function.

Definition 3.4. [minimal separator]
δ is called a *minimal separator* for $\langle A, B\rangle$ if δ is a separator for $\langle A, B\rangle$ and $w(\delta) = \mathit{MIN}\ \{w(\delta') : \delta'$ is a separator for $\langle A, B\rangle\}$. □

Now we are ready to formally state our main lower bound theorem on the size of $\text{FO}^3(<)$-formulas:

Theorem 3.5. [main lower bound theorem]
If ψ is an $\text{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-formula, A and B are sets of interpretations such that $A \models \psi$ and $B \models \neg\psi$, and δ is a minimal separator for $\langle A, B\rangle$, then

$$||\psi|| \geqslant \tfrac{1}{2} \cdot w(\delta)\,.$$

□

Before giving details on the proof of Theorem 3.5, let us first point out its following easy consequence:

Corollary 3.6. *Let $n > m \geqslant 0$. The two linear orders $\mathcal{A}_m$ and $\mathcal{A}_n$ (with universe $\{0,..,m\}$ and $\{0,..,n\}$, respectively) cannot be distinguished by an $\text{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-sentence of size $< \frac{1}{2}\sqrt{m}$.* □

Proof: Let ψ be an $\text{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-sentence with $\mathcal{A}_m \models \psi$ and $\mathcal{A}_n \models \neg\psi$. Let α be the assignment that maps each of the variables x, y, and z to the value 0. Consider the mapping $\delta_m : \mathcal{P}_2(\{\mathit{min}, \mathit{max}, x, y, z\}) \to \mathbb{N}$ with $\delta_m(p) = m$ if $p = \{\mathit{min}, \mathit{max}\}$, and $\delta_m(p) = 0$ otherwise. It is straightforward to check that $w(\delta_m) = \sqrt{m}$ and that δ_m is a *minimal separator* for $\langle(\mathcal{A}_m, \alpha), (\mathcal{A}_n, \alpha)\rangle$. From Theorem 3.5 we therefore obtain that $||\psi|| \geqslant \frac{1}{2}\cdot w(\delta_m) = \frac{1}{2}\cdot\sqrt{m}$. This completes the proof of Corollary 3.6. ■

To prove Theorem 3.5 we need a series of intermediate results, as well as the notion of an *extended syntax tree* of a formula, which is a syntax tree where each node carries an additional label containing information about sets of interpretations satisfying, respectively, not satisfying, the associated subformula. More precisely, every node v of the extended syntax tree carries an *interpretation label* $\mathit{il}(v)$ which consists of a pair $\langle A, B\rangle$ of sets of interpretations such that every interpretation in A, but no interpretation in B, satisfies the subformula represented by the subtree rooted at node v. Basically, such an extended syntax tree corresponds to a game tree that is constructed by the two players of the *Adler-Immerman game* (cf., [1]).

For proving Theorem 3.5 we consider an extended syntax tree $\mathcal{T}$ of the given formula ψ. We define a weight function on the nodes of $\mathcal{T}$ by defining the weight $w(v)$ of each node v of $\mathcal{T}$ to be the weight of a *minimal separator* for $\mathit{il}(v)$. Afterwards – and this is the main technical difficulty – we show that the weight of each node v is bounded (from above) by the weights of v's children. This, in turn, enables us to prove a lower bound on the number of nodes in $\mathcal{T}$ which depends on the weight of the root node.

3.2. Formal Proof of Theorem 3.5.

We start with the formal definition of extended syntax trees.

Definition 3.7. [extended syntax tree]
Let ψ be an $\text{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-formula, let A and B be sets of interpretations such that $A \models \psi$ and $B \models \neg\psi$. By induction on the construction of ψ we define an *extended syntax tree* $\mathcal{T}_\psi^{\langle A,B\rangle}$ as follows:

- If ψ is an atomic formula, then $\mathcal{T}_\psi^{\langle A,B\rangle}$ consists of a single node v that has a *syntax label* $\mathit{sl}(v) := \psi$ and an *interpretation label* $\mathit{il}(v) := \langle A, B\rangle$.
- If ψ is of the form $\neg\psi_1$, then $\mathcal{T}_\psi^{\langle A,B\rangle}$ has a root node v with $\mathit{sl}(v) := \neg$ and $\mathit{il}(v) := \langle A, B\rangle$. The unique child of v is the root of $\mathcal{T}_{\psi_1}^{\langle B,A\rangle}$. Note that $B \models \psi_1$ and $A \models \neg\psi_1$.
- If ψ is of the form $\psi_1 \vee \psi_2$, then $\mathcal{T}_\psi^{\langle A,B\rangle}$ has a root node v with $\mathit{sl}(v) := \vee$ and $\mathit{il}(v) := \langle A, B\rangle$.
  The first child of v is the root of $\mathcal{T}_{\psi_1}^{\langle A_1,B\rangle}$. The second child of v is the root of $\mathcal{T}_{\psi_2}^{\langle A_2,B\rangle}$, where, for $i \in \{1, 2\}$, $A_i = \{(\mathcal{A}, \alpha) \in A : (\mathcal{A}, \alpha) \models \psi_i\}$.
  Note that $A = A_1 \cup A_2$, $A_i \models \psi_i$, and $B \models \neg\psi_i$.
- If ψ is of the form $\psi_1 \wedge \psi_2$, then $\mathcal{T}_\psi^{\langle A,B\rangle}$ has a root node v with $\mathit{sl}(v) := \wedge$ and $\mathit{il}(v) := \langle A, B\rangle$.
  The first child of v is the root of $\mathcal{T}_{\psi_1}^{\langle A,B_1\rangle}$. The second child of v is the root of $\mathcal{T}_{\psi_2}^{\langle A,B_2\rangle}$, where, for $i \in \{1, 2\}$, $B_i = \{(\mathcal{B}, \beta) \in B : (\mathcal{B}, \beta) \not\models \psi_i\}$.
  Note that $B = B_1 \cup B_2$, $A \models \psi_i$, and $B_i \models \neg\psi_i$.
- If ψ is of the form $\exists u\, \psi_1$, for a variable $u \in \{x, y, z\}$, then $\mathcal{T}_\psi^{\langle A,B\rangle}$ has a root node v with $\mathit{sl}(v) := \exists u$ and $\mathit{il}(v) := \langle A, B\rangle$. The unique child of v is the root of $\mathcal{T}_{\psi_1}^{\langle A_1,B_1\rangle}$, where $B_1 := \{(\mathcal{B}, \beta[\frac{b}{u}]) : (\mathcal{B}, \beta) \in B,\ b \in \mathcal{U}^{\mathcal{B}}\}$, and A_1 is chosen as follows: For every $(\mathcal{A}, \alpha) \in A$ fix an element $a \in \mathcal{U}^{\mathcal{A}}$ such that $(\mathcal{A}, \alpha[\frac{a}{u}]) \models \psi_1$, and let $A_1 := \{(\mathcal{A}, \alpha[\frac{a}{u}]) : (\mathcal{A}, \alpha) \in A\}$.
  Note that $A_1 \models \psi_1$ and $B_1 \models \neg\psi_1$.
- If ψ is of the form $\forall u\, \psi_1$, for a variable $u \in \{x, y, z\}$, then $\mathcal{T}_\psi^{\langle A,B\rangle}$ has a root node v with $\mathit{sl}(v) := \forall u$ and $\mathit{il}(v) := \langle A, B\rangle$. The unique child of v is the root of $\mathcal{T}_{\psi_1}^{\langle A_1,B_1\rangle}$, where $A_1 := \{(\mathcal{A}, \alpha[\frac{a}{u}]) : (\mathcal{A}, \alpha) \in A,\ a \in \mathcal{U}^{\mathcal{A}}\}$, and B_1 is chosen as follows: For every $(\mathcal{B}, \beta) \in B$ fix an element $b \in \mathcal{U}^{\mathcal{B}}$ such that $(\mathcal{B}, \beta[\frac{b}{u}]) \models \neg\psi_1$, and let $B_1 := \{(\mathcal{B}, \alpha[\frac{b}{u}]) : (\mathcal{B}, \beta) \in B\}$.
  Note that $A_1 \models \psi_1$ and $B_1 \models \neg\psi_1$. □

The following is the main technical result necessary for our proof of Theorem 3.5.

Lemma 3.8. *Let ψ be an* $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$*-formula, let A and B be sets of interpretations such that $A \models \psi$ and $B \models \neg\psi$, and let $\mathcal{T}$ be an extended syntax tree $\mathcal{T}_\psi^{\langle A,B\rangle}$.*
For every node v of $\mathcal{T}$ the following is true, where δ is a minimal separator for $il(v)$:

(a) If v is a leaf *then $w(\delta) \leqslant 1$.*

(b) If v has 2 children v_1 and v_2, and δ_i is a minimal separator for $il(v_i)$, for $i \in \{1,2\}$, then $w(\delta) \leqslant w(\delta_1) + w(\delta_2)$.

(c) If v has exactly one child v_1, and δ_1 is a minimal separator for $il(v_1)$, then $w(\delta) \leqslant w(\delta_1) + 2$. □

The proof of Lemma 3.8 is given in Section 3.3 below.

For a binary tree $\mathcal{T}$ we write $||\mathcal{T}||$ to denote the number of nodes of $\mathcal{T}$. For the proof of Theorem 3.5 we also need the following easy observation.

Lemma 3.9. *Let $\mathcal{T}$ be a finite binary tree where each node v is equipped with a* weight $w(v) \geqslant 0$ *such that the following is true:*

(a) If v is a leaf then $w(v) \leqslant 1$.

(b) If v has 2 children v_1 and v_2, then $w(v) \leqslant w(v_1) + w(v_2)$.

(c) If v has exactly one child v_1, then $w(v) \leqslant w(v_1) + 2$.

Then, $||\mathcal{T}|| \geqslant \frac{1}{2} \cdot w(r)$, where r is the root of $\mathcal{T}$. □

The (straightforward) proof is given in Appendix A.
Using Lemma 3.8 and 3.9, we are ready for the

Proof of Theorem 3.5:
We are given an $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-formula ψ and sets A and B of interpretations such that $A \models \psi$ and $B \models \neg\psi$. Let $\mathcal{T}$ be an extended syntax tree $\mathcal{T}_\psi^{\langle A,B\rangle}$.

We equip each node v of $\mathcal{T}$ with a *weight* $w(v) := w(\delta_v)$, where δ_v is a minimal separator for $il(v)$. From Lemma 3.8 we obtain that the preconditions of Lemma 3.9 are satisfied. Therefore, $||\mathcal{T}|| \geqslant \frac{1}{2} \cdot w(r)$, where r is the root of $\mathcal{T}$, i.e., $w(r) = w(\delta)$, for a minimal separator δ for $il(r) = \langle A, B\rangle$.
From Definition 3.7 it should be obvious that $||\psi|| = ||\mathcal{T}||$. Therefore, the proof of Theorem 3.5 is complete. ■

3.3. Proof of Lemma 3.8.

We partition the proof of Lemma 3.8 into proofs for the parts *(a)*, *(b)*, and *(c)*, where part *(c)* turns out to be the most elaborate.

According to the assumptions of Lemma 3.8 we are given an $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-formula ψ and sets A and B of interpretations such that $A \models \psi$ and $B \models \neg\psi$. Furthermore, we are given an extended syntax tree $\mathcal{T} = \mathcal{T}_\psi^{\langle A,B\rangle}$. Throughout the remainder of this section, $\mathcal{T}$ will always denote this particular syntax tree.

The proof of part *(a)* of Lemma 3.8 is straightforward; it is given in Appendix A.
The essential step in the proof of part *(b)* of Lemma 3.8 is the following easy lemma.

Lemma 3.10. *Let v be a node of $\mathcal{T}$ that has two children v_1 and v_2. Let δ_1 and δ_2 be separators for $il(v_1)$ and $il(v_2)$, respectively. Let $\tilde{\delta}$ be the potential separator defined on every $p \in \mathcal{P}_2(\{\mathit{min}, \mathit{max}, x, y, z\})$ via*

$$\tilde{\delta}(p) := \delta_1(p) + \delta_2(p).$$

Then, $\tilde{\delta}$ is a separator *for $il(v)$.* □

The (straightforward) proof is given in Appendix A.
Using Lemma 3.10, the proof of part *(b)* of Lemma 3.8 is straightforward.
An essential step in the proof of part *(c)* of Lemma 3.8 is the following lemma.

Lemma 3.11. *Let v be a node of $\mathcal{T}$ that has syntax-label $sl(v) = \mathsf{Q}u$, for $\mathsf{Q} \in \{\exists, \forall\}$ and $u \in \{x, y, z\}$. Let δ_1 be a separator for $il(v_1)$, where v_1 is the unique child of v in $\mathcal{T}$. Let $\tilde{\delta}$ be the potential separator defined via*

- $\tilde{\delta}(\{u, u'\}) := 0$, *for all* $u' \in \{\mathit{min}, \mathit{max}, x, y, z\} \setminus \{u\}$,
- $\tilde{\delta}(\{\mathit{min}, \mathit{max}\}) := \mathit{MAX}\ \{\ \delta_1(\{\mathit{min}, \mathit{max}\}),\ \delta_1(\{\mathit{min}, u\}) + \delta_1(\{u, \mathit{max}\}) + 1\ \}$,

and for all u', u'' such that $\{x, y, z\} = \{u, u', u''\}$ and all $m \in \{\mathit{min}, \mathit{max}\}$,

- $\tilde{\delta}(\{u', u''\}) := \mathit{MAX}\ \{\ \delta_1(\{u', u''\}),\ \delta_1(\{u', u\}) + \delta_1(\{u, u''\}) + 1\ \}$,
- $\tilde{\delta}(\{m, u'\}) := \mathit{MAX}\ \{\ \delta_1(\{m, u'\}),\ \delta_1(\{m, u\}) + \delta_1(\{u, u'\}) + 1\ \}$.

Then, $\tilde{\delta}$ is a separator *for $il(v)$.* □

Proof: We only consider the case where $\mathsf{Q}u = \exists z$. All other cases $\mathsf{Q} \in \{\exists, \forall\}$ and $u \in \{x, y, z\}$ follow by symmetry.

Let $\langle A, B\rangle := il(v)$. We need to show that $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J}\rangle$, for all $\mathcal{I} \in A$ and $\mathcal{J} \in B$. Let therefore $\mathcal{I} := (\mathcal{A}, \alpha) \in A$ and $\mathcal{J} := (\mathcal{B}, \beta) \in B$ be fixed for the remainder of this proof. The aim is to show that $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J}\rangle$.

Since $sl(v) = \exists z$, we know from Definition 3.7 that $il(v_1) = \langle A_1, B_1\rangle$, where B_1 contains the interpretations $(\mathcal{B}, \beta[\frac{b}{z}])$, for all $b \in \mathcal{U}^{\mathcal{B}}$, and A_1 contains an interpretation $(\mathcal{A}, \alpha[\frac{a}{z}])$, for a particular $a \in \mathcal{U}^{\mathcal{A}}$. We define $\alpha_a := \alpha[\frac{a}{z}]$, $\mathcal{I}_a := (\mathcal{A}, \alpha_a)$, and for every $b \in \mathcal{U}^{\mathcal{B}}$, $\beta_b := \beta[\frac{b}{z}]$ and $\mathcal{J}_b := (\mathcal{B}, \beta_b)$.

From the fact that δ_1 is a separator for $sl(v_1)$, we in particular know, for every $b \in \mathcal{U}^{\mathcal{B}}$, that δ_1 is a separator for $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$. I.e., we know the following:

For every $b \in \mathcal{U}^{\mathcal{B}}$ there are $u_b, u'_b \in \{min, max, x, y, z\}$ with $u_b \neq u'_b$, such that *$\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by* $\delta_1(\{u_b, u'_b\})$, i.e., $\delta_1(\{u_b, u'_b\}) \geqslant 1$ and

$(1)_b$: $<\text{-}type\big(\alpha_a(u_b), \alpha_a(u'_b)\big) \neq <\text{-}type\big(\beta_b(u_b), \beta_b(u'_b)\big)$, or

$(2)_b$: $\delta_1\big(\{u_b, u'_b\}\big) \geqslant MIN\,\big\{|diff\big(\alpha_a(u_b), \alpha_a(u'_b)\big)|\,,\ |diff\big(\beta_b(u_b), \beta_b(u'_b)\big)|\big\}$ and $diff\big(\alpha_a(u_b), \alpha_a(u'_b)\big) \neq diff\big(\beta_b(u_b), \beta_b(u'_b)\big)$.

In what follows we will prove a series of claims which ensure that $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$. We start with

Claim 1. *If there is a $b \in \mathcal{U}^{\mathcal{B}}$ such that $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by $\delta_1(\{u_b, u'_b\})$ with $z \notin \{u_b, u'_b\}$, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

Proof: By definition of $\tilde{\delta}$ we have $\tilde{\delta}(\{u_b, u'_b\}) \geqslant \delta_1(\{u_b, u'_b\})$. Therefore $(1)_b$ and $(2)_b$ imply that $\tilde{\delta}$ is a separator for $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$, as well as for $\langle \mathcal{I}, \mathcal{J} \rangle$.
This completes the proof of Claim 1. ■

Due to Claim 1 it henceforth suffices to assume that for no $b \in \mathcal{U}^{\mathcal{B}}$, $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by $\delta_1(\{u_b, u'_b\})$ with $z \notin \{u_b, u'_b\}$. I.e., we assume that for every $b \in \mathcal{U}^{\mathcal{B}}$, $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by $\delta_1(\{min, z\})$, $\delta_1(\{z, max\})$, $\delta_1(\{x, z\})$, or $\delta_1(\{y, z\})$.

Claim 2. *If $a = \alpha(u)$ for some $u \in \{min, max, x, y\}$, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

Proof: Choose $b := \beta(u)$. Therefore, $\alpha_a(z) = a = \alpha(u) = \alpha_a(u)$ and $\beta_b(z) = b = \beta(u) = \beta_b(u)$.

We know that $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by $\delta_1(\{z, u'\})$, for some $u' \in \{min, max, x, y\}$. Furthermore, since $\alpha_a(z) = \alpha_a(u)$ and $\beta_b(z) = \beta_b(u)$, we have $u' \neq u$.

By definition of $\tilde{\delta}$ we know that $\tilde{\delta}(\{u, u'\}) \geqslant \delta_1(\{z, u'\})$. Therefore, $\tilde{\delta}$ is a separator for $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ as well as for $\langle \mathcal{I}, \mathcal{J} \rangle$.
This completes the proof of Claim 2. ■

Due to Claim 2 it henceforth suffices to assume that, $a \neq \alpha(u)$, for all $u \in \{min, max, x, y\}$.

Claim 3. *If $\delta_1(\{min, z\}) \geqslant diff(a, min^{\mathcal{A}})$, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

Proof: We distinguish between two cases.
<u>*Case 1:*</u> $diff(max^{\mathcal{B}}, min^{\mathcal{B}}) \geqslant diff(a, min^{\mathcal{A}})$.
In this case we can choose $b \in \mathcal{U}^{\mathcal{B}}$ with $diff(b, min^{\mathcal{B}}) = diff(a, min^{\mathcal{A}})$ (simply via $b := a$). Obviously, $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is not separated by $\delta_1(\{min, z\})$. However, we know that $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by $\delta_1(\{z, u'\})$, for some $u' \in \{x, y, max\}$.
Since $diff(a, min^{\mathcal{A}}) + \delta_1(\{z, u'\}) \leqslant \delta_1(\{min, z\}) + \delta_1(\{z, u'\}) \leqslant \tilde{\delta}(\{min, u'\})$, it is straightforward to see that $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$, and also $\langle \mathcal{I}, \mathcal{J} \rangle$, is separated by $\tilde{\delta}(\{min, u'\})$. I.e., $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.
<u>*Case 2:*</u> $diff(max^{\mathcal{B}}, min^{\mathcal{B}}) < diff(a, min^{\mathcal{A}})$.
Since $\tilde{\delta}(\{min, max\}) \geqslant \delta_1(\{min, z\}) \geqslant diff(a, min^{\mathcal{A}}) > diff(max^{\mathcal{B}}, min^{\mathcal{B}})$, we know that $\langle \mathcal{I}, \mathcal{J} \rangle$ is separated by $\tilde{\delta}(\{min, max\})$. I.e., $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.
This completes the proof of Claim 3. ■

By symmetry we also obtain the following

Claim 4. *If $\delta_1(\{z, max\}) \geqslant diff(max^{\mathcal{A}}, a)$, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

In a similar way, we can also show the following

Claim 5. *If $\delta_1(\{x, z\}) \geqslant |diff(\alpha(x), a)|$, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

Proof: We distinguish between three cases.
<u>*Case 1:*</u> *There is a $b \in \mathcal{U}^{\mathcal{B}}$ such that $diff(\beta(x), b) = diff(\alpha(x), a)$.*
Obviously, $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is not separated by $\delta_1(\{x, z\})$. However, we know that $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$ is separated by $\delta_1(\{z, u'\})$, for some $u' \in \{min, y, max\}$.
Since $|diff(\alpha(x), a)| + \delta_1(\{z, u'\}) \leqslant \delta_1(\{x, z\}) + \delta_1(\{z, u'\}) \leqslant \tilde{\delta}(\{x, u'\})$, it is straightforward to see that $\langle \mathcal{I}_a, \mathcal{J}_b \rangle$, and also $\langle \mathcal{I}, \mathcal{J} \rangle$, is separated by $\tilde{\delta}(\{x, u'\})$. I.e., $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.
<u>*Case 2:*</u> *$\alpha(x) \leqslant a$ and $diff(max^{\mathcal{B}}, \beta(x)) < diff(a, \alpha(x))$.*
Since $\tilde{\delta}(\{x, max\}) \geqslant \delta_1(\{x, z\}) \geqslant diff(a, \alpha(x)) > diff(max^{\mathcal{B}}, \beta(x))$, we know that $\langle \mathcal{I}, \mathcal{J} \rangle$ is separated by $\tilde{\delta}(\{x, max\})$. I.e., $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.
<u>*Case 3:*</u> *$a < \alpha(x)$ and $diff(\beta(x), min^{\mathcal{B}}) < diff(\alpha(x), a)$.*
This case is analogous to Case 2.

Now the proof of Claim 5 is complete, because one of the three cases above must apply. ■

By symmetry we also obtain the following

Claim 6. *If $\delta_1(\{y, z\}) \geqslant |diff(\alpha(y), a)|$, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

Finally, we show the following

Claim 7. *If none of the assumptions of the Claims 1–6 is satisfied, then $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J} \rangle$.* □

Proof: We assume w.l.o.g. that $min^{\mathcal{A}} \leqslant \alpha(x) \leqslant \alpha(y) \leqslant max^{\mathcal{A}}$.
Since Claim 1 does not apply, we know that also $min^{\mathcal{B}} \leqslant \beta(x) \leqslant \beta(y) \leqslant max^{\mathcal{B}}$. Since Claims 2–6 do not apply, we furthermore know that

1. $|\mathit{diff}(\alpha(u'),a)| > \delta_1(\{u',z\})$, for all $u' \in \{\mathit{min},\mathit{max},x,y\}$, and

2. $\mathit{min}^{\mathcal{A}} < a < \alpha(x)$ or $\alpha(x) < a < \alpha(y)$ or $\alpha(y) < a < \mathit{max}^{\mathcal{A}}$.

We distinguish between different cases, depending on the particular interval that a belongs to.

Case 1: $\mathit{min}^{\mathcal{A}} < a < \alpha(x)$.
Case 1.1: $\mathit{diff}(\beta(x),\mathit{min}^{\mathcal{B}}) \leqslant \delta_1(\{\mathit{min},z\})$.
By definition of $\tilde{\delta}$ we have $\mathit{diff}(\beta(x),\mathit{min}^{\mathcal{B}}) \leqslant \tilde{\delta}(\{\mathit{min},x\})$. Since $\mathit{diff}(\alpha(x),\mathit{min}^{\mathcal{A}}) > \mathit{diff}(a,\mathit{min}^{\mathcal{A}}) > \delta_1(\{\mathit{min},z\}) \geqslant \mathit{diff}(\beta(x),\mathit{min}^{\mathcal{B}})$, we therefore know that $\langle\mathcal{I},\mathcal{J}\rangle$ is separated by $\tilde{\delta}(\{\mathit{min},z\})$. I.e., $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$.
Case 1.2: $\mathit{diff}(\beta(x),\mathit{min}^{\mathcal{B}}) > \delta_1(\{\mathit{min},z\})$.
In this case we can choose $b \leqslant \beta(x)$ such that $\mathit{diff}(b,\mathit{min}^{\mathcal{B}}) = \delta_1(\{\mathit{min},z\}) + 1$. Then, $\langle\mathcal{I}_a,\mathcal{J}_b\rangle$ is not separated by $\delta_1(\{\mathit{min},z\})$. However, we know that $\langle\mathcal{I}_a,\mathcal{J}_b\rangle$ is separated by $\delta_1(\{z,u'\})$, for some $u' \in \{x,y,\mathit{max}\}$. From the assumptions of Claim 7 we know that $\mathit{diff}(\alpha(u'),a) > \delta_1(\{z,u'\})$. Hence we must have that $\mathit{diff}(\beta(u'),b) \leqslant \delta_1(\{z,u'\})$. Therefore, $\mathit{diff}(\beta(u'),\mathit{min}^{\mathcal{B}}) = \mathit{diff}(\beta(u'),b) + \mathit{diff}(b,\mathit{min}^{\mathcal{B}}) \leqslant \delta_1(\{z,u'\}) + \delta_1(\{\mathit{min},z\}) + 1 \leqslant \tilde{\delta}(\{\mathit{min},u'\})$. Since $\mathit{diff}(\alpha(u'),\mathit{min}^{\mathcal{A}}) = \mathit{diff}(\alpha(u'),a) + \mathit{diff}(a,\mathit{min}^{\mathcal{A}}) > \mathit{diff}(\beta(u'),\mathit{min}^{\mathcal{B}})$, we hence obtain that $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$.

Case 2: $\alpha(y) < a < \mathit{max}^{\mathcal{A}}$.
This case is analogous to Case 1.

Case 3: $\alpha(x) < a < \alpha(y)$.
Case 3.1: $\mathit{diff}(\beta(y),\beta(x)) \leqslant \delta_1(\{x,z\})$.
By definition of $\tilde{\delta}$ we have $\mathit{diff}(\beta(y),\beta(x)) \leqslant \tilde{\delta}(\{x,y\})$. Since $\mathit{diff}(\alpha(y),\alpha(x)) > \mathit{diff}(a,\alpha(x)) > \delta_1(\{x,z\}) \geqslant \mathit{diff}(\beta(y),\beta(x))$, we therefore know that $\langle\mathcal{I},\mathcal{J}\rangle$ is separated by $\tilde{\delta}(\{x,y\})$. I.e., $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$.
Case 3.2: $\mathit{diff}(\beta(y),\beta(x)) > \delta_1(\{x,z\})$.
In this case we can choose b with $\beta(x) < b \leqslant \beta(y)$ such that $\mathit{diff}(b,\beta(x)) = \delta_1(\{x,z\}) + 1$. Then, $\langle\mathcal{I}_a,\mathcal{J}_b\rangle$ is not separated by $\delta_1(\{x,z\})$. However, we know that $\langle\mathcal{I}_a,\mathcal{J}_b\rangle$ is separated by $\delta_1(\{z,u'\})$, for some $u' \in \{\mathit{min},y,\mathit{max}\}$. From the assumptions of Claim 7 we know that $|\mathit{diff}(a,\alpha(u'))| > \delta_1(\{z,u'\})$. Hence we must have that $|\mathit{diff}(b,\beta(u'))| \leqslant \delta_1(\{z,u'\})$. We now distinguish between the cases where u' can be chosen from $\{y,\mathit{max}\}$, on the one hand, and where u' must be chosen as min, on the other hand.
Case 3.2.1: $u' \in \{y,\mathit{max}\}$.
In this case, $\mathit{diff}(\beta(u'),\beta(x)) = \mathit{diff}(\beta(u'),b) + \mathit{diff}(b,\beta(x)) \leqslant \delta_1(\{z,u'\}) + \delta_1(\{x,z\}) + 1 \leqslant \tilde{\delta}(\{x,u'\})$. Since $\mathit{diff}(\alpha(u'),\alpha(x)) = \mathit{diff}(\alpha(u'),a) + \mathit{diff}(a,\alpha(x)) > \mathit{diff}(\beta(u'),\beta(x))$, we hence obtain that $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$.
Case 3.2.2: $u' \notin \{y,\mathit{max}\}$.
In this case, $\langle\mathcal{I}_a,\mathcal{J}_b\rangle$ is separated by $\delta_1(\{\mathit{min},z\})$, and we may assume that it is neither separated by $\delta_1(\{z,y\})$ nor by $\delta_1(\{z,\mathit{max}\})$ nor by $\delta_1(\{x,z\})$. In particular, we must have that $\mathit{diff}(\beta(y),b) \geqslant \delta_1(\{z,y\}) + 1$. Therefore, for every b' with $b \leqslant b' < \beta(y) - \delta_1(\{z,y\})$, the following is true: $\langle\mathcal{I}_a,\mathcal{J}_{b'}\rangle$ is neither separated by $\delta_1(\{x,z\})$ nor by $\delta_1(\{z,y\})$, but, consequently, by $\delta_1(\{\mathit{min},z\})$ or by $\delta_1(\{z,\mathit{max}\})$. Let b_1 be the largest such b' for which $\langle\mathcal{I}_a,\mathcal{J}_{b'}\rangle$ is separated by $\delta_1(\{\mathit{min},z\})$. In particular, $\mathit{diff}(b_1,\mathit{min}^{\mathcal{B}}) \leqslant \delta_1(\{\mathit{min},z\})$.
Case 3.2.2.1: $\mathit{diff}(\beta(y),b_1{+}1) \leqslant \delta_1(\{z,y\})$.
In this case we know that $\mathit{diff}(\beta(y),\mathit{min}^{\mathcal{B}}) \leqslant \delta_1(\{z,y\}) + 1 + \delta_1(\{\mathit{min},z\}) \leqslant \tilde{\delta}(\{\mathit{min},y\})$. Furthermore, $\mathit{diff}(\alpha(y),\mathit{min}^{\mathcal{A}}) = \mathit{diff}(\alpha(y),a) + \mathit{diff}(a,\mathit{min}^{\mathcal{A}}) \geqslant \delta_1(\{z,y\}) + 1 + \delta_1(\{\mathit{min},z\}) + 1$. Therefore $\mathit{diff}(\alpha(y),\mathit{min}^{\mathcal{A}}) \neq \mathit{diff}(\beta(y),\mathit{min}^{\mathcal{B}})$, and $\langle\mathcal{I},\mathcal{J}\rangle$ is separated by $\tilde{\delta}(\{\mathit{min},y\})$. I.e., $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$.
Case 3.2.2.2: $\mathit{diff}(\beta(y),b_1{+}1) > \delta_1(\{z,y\})$.
In this case we know (by the maximal choice of b_1) that $\langle\mathcal{I}_a,\mathcal{J}_{b_1+1}\rangle$ must be separated by $\delta_1(\{z,\mathit{max}\})$. In particular, $\mathit{diff}(\mathit{max}^{\mathcal{B}},b_1{+}1) \leqslant \delta_1(z,\mathit{max})$. Therefore, $\mathit{diff}(\mathit{max}^{\mathcal{B}},\mathit{min}^{\mathcal{B}}) \leqslant \delta_1(\{z,\mathit{max}\}) + 1 + \delta_1(\{\mathit{min},z\}) \leqslant \tilde{\delta}(\{\mathit{min},\mathit{max}\})$. Furthermore, $\mathit{diff}(\mathit{max}^{\mathcal{A}},\mathit{min}^{\mathcal{A}}) \geqslant \mathit{diff}(\mathit{max}^{\mathcal{A}},a) + \mathit{diff}(a,\mathit{min}^{\mathcal{A}}) \geqslant \delta_1(\{z,\mathit{max}\}) + 1 + \delta_1(\{\mathit{min},z\}) + 1$. Therefore $\mathit{diff}(\mathit{max}^{\mathcal{A}},\mathit{min}^{\mathcal{A}}) \neq \mathit{diff}(\mathit{max}^{\mathcal{B}},\mathit{min}^{\mathcal{B}})$, and $\langle\mathcal{I},\mathcal{J}\rangle$ is separated by $\tilde{\delta}(\{\mathit{min},\mathit{max}\})$. I.e., $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$.

We now have shown that $\tilde{\delta}$ is a separator for $\langle\mathcal{I},\mathcal{J}\rangle$, if Case 3 applies.
Together with the Cases 1 and 2 we therefore obtain that the proof of Claim 7 is complete. ■

Since at least one of the Claims 1–7 must apply, the proof of Lemma 3.11 finally is complete. ■

Proof of part *(c)* of Lemma 3.8:
Let v be a node of $\mathcal{T}$ that has exactly one child v_1. Let δ be a minimal separator for $\mathit{il}(v)$, and let δ_1 be a minimal separator for $\mathit{il}(v_1) =: \langle A_1,B_1\rangle$. Our aim is to show that $w(\delta) \leqslant w(\delta_1) + 2$.

From Definition 3.7 we know that either $\mathit{sl}(v) = \neg$ or $\mathit{sl}(v) = \mathsf{Q}u$, for some $\mathsf{Q} \in \{\exists,\forall\}$ and $u \in \{x,y,z\}$.

Case 1: $\mathit{sl}(v) = \neg$
In this case we know from Definition 3.7 that $\mathit{il}(v) = \langle B_1,A_1\rangle$. Therefore, δ_1 also is a (minimal) separator for $\mathit{il}(v)$. In particular, $w(\delta) = w(\delta_1) \leqslant w(\delta_1) + 2$.

Case 2: $\mathit{sl}(v) = \mathsf{Q}u$
In this case let $\tilde{\delta}$ be the *separator for $\mathit{il}(v)$* defined in Lemma 3.11. Since δ is a *minimal* separator for $\mathit{il}(v)$, it suffices to show that $w(\tilde{\delta}) \leqslant w(\delta_1) + 2$.

Let u', u'' be chosen such that $\{x,y,z\} = \{u,u',u''\}$. Using Definition 3.3 and the particular choice of $\tilde{\delta}$, it is straightforward to see that

$$c(\tilde{\delta}) \quad = \quad \tilde{\delta}(\{u',u''\}) \quad \leqslant \quad c(\delta_1)+1 \qquad (1)$$

and that

$$\tilde{\delta}(\{min, max\}) \;\leqslant\; b(\delta_1)+1\,. \qquad (2)$$

Furthermore, for arbitrary $\tilde{u}, \tilde{u}' \in \{x,y,z\}$ we have

$$\tilde{\delta}(\{min,\tilde{u}\}) + \tilde{\delta}(\{\tilde{u}', max\}) \;\leqslant\; b(\delta_1)+2c(\delta_1)+2\,, \qquad (3)$$

which can be seen as follows: If $\tilde{u} = u$ or $\tilde{u}' = u$, then $\tilde{\delta}(\{min,\tilde{u}\}) = 0$ or $\tilde{\delta}(\{\tilde{u}',max\}) = 0$. Consequently, $\tilde{\delta}(\{min,\tilde{u}\}) + \tilde{\delta}(\{\tilde{u}',max\}) \leqslant MAX\ \{b(\delta_1),\ b(\delta_1) + c(\delta_1)+1\}$. If $\tilde{u}$ and $\tilde{u}'$ both are different from u, then

$$\tilde{\delta}(\{min,\tilde{u}\}) \;=\; MAX\,\big\{\,\delta_1(\{min,\tilde{u}\}),\\ \delta_1(\{min,u\})+\delta_1(\{u,\tilde{u}\})+1\,\big\}$$

and

$$\tilde{\delta}(\{\tilde{u}',max\}) \;=\; MAX\,\big\{\,\delta_1(\{\tilde{u}',max\}),\\ \delta_1(\{u,max\})+\delta_1(\{u,\tilde{u}'\})+1\,\big\}\,.$$

Therefore,

$$\tilde{\delta}(\{min,\tilde{u}\}) + \tilde{\delta}(\{\tilde{u}',max\}) \;\leqslant\; MAX \left\{ \begin{array}{l} \delta_1(\{min,\tilde{u}\}) + \delta_1(\{\tilde{u}',max\}),\\ \delta_1(\{min,\tilde{u}\}) + \delta_1(\{u,max\}) + \delta_1(\{u,\tilde{u}'\}) + 1,\\ \delta_1(\{min,u\}) + \delta_1(\{u,\tilde{u}\}) + 1 + \delta_1(\{\tilde{u}',max\}),\\ \delta_1(\{min,u\}) + \delta_1(\{u,\tilde{u}\}) + 1 + \\ \qquad \delta_1(\{u,max\}) + \delta_1(\{u,\tilde{u}'\}) + 1 \end{array} \right\}$$

which, in turn, is less than or equal to

$$MAX\,\big\{\,b(\delta_1),\;\; b(\delta_1)+c(\delta_1)+1,\;\; b(\delta_1)+2c(\delta_1)+2\,\big\}\,.$$

I.e., we have shown that (3) is valid.
From (2) and (3) we obtain

$$b(\tilde{\delta}) \quad \leqslant \quad b(\delta_1)+2c(\delta_1)+2\,. \qquad (4)$$

From (1) and (4) we conclude that

$$\begin{array}{rcl} w(\tilde{\delta})^2 & = & c(\tilde{\delta})^2 + b(\tilde{\delta}) \\ & \leqslant & \big(c(\delta_1)+1\big)^2 + b(\delta_1) + 2c(\delta_1) + 2 \\ & = & c(\delta_1)^2 + b(\delta_1) + 4c(\delta_1) + 3 \\ & \leqslant & w(\delta_1)^2 + 4w(\delta_1) + 4 \\ & = & \big(w(\delta_1)+2\big)^2\,. \end{array}$$

Therefore, $w(\tilde{\delta}) \leqslant w(\delta_1)+2$.
This completes the proof of part *(c)* of Lemma 3.8. ■

Remark 3.12. From the above proof it becomes clear, why Definition 3.3 fixes the *weight* of a separator by using the $\sqrt{\ }$-function. Let us consider the, at first glance, more straightforward weight function $\hat{w}(\delta) := MAX\ \{c(\delta),\ b(\delta)\}$. In the proof of part *(c)* of Lemma 3.8 we then obtain from the items (1) and (4) that $\hat{w}(\delta) \leqslant 2c(\delta_1) + b(\delta_1) + 2 \leqslant 3\hat{w}(\delta_1) + 2$. Therefore, a modified version of Lemma 3.9, where item (c) is replaced by the condition "*If v has exactly one child v_1, then $w(v) \leqslant 3w(v_1) + 2$*", leads to a (much weaker) bound of the form $||\mathcal{T}|| \geqslant c \cdot \lg(w(v))$. This, in turn, leads to a weaker version of Theorem 3.5, stating that $||\psi|| \geqslant c \cdot \lg\big(\hat{w}(\delta)\big)$. However, this bound can already be proven by a conventional Ehrenfeucht-Fraïssé game and does not only apply for $\mathrm{FO}^3(<)$-formulas but for FO($<$)-formulas in general and therefore is of no use for comparing the succinctness of FO^3 and FO. □

4. FO^2 vs. FO^3

As a first application, Theorem 3.5 allows us to translate every FO^3-sentence ψ into an FO^2-sentence χ that is equivalent to ψ on linear orders and that has size polynomial in the size of ψ. To show this, we use the following easy lemmas.

Lemma 4.1. *Let φ be an* FO($<$, *Succ*, *min*, *max*)*-sentence, and let d be the quantifier depth of φ. For all $N \geqslant 2^{d+1}$, $\mathcal{A}_N \models \varphi$ if, and only if, $\mathcal{A}_{2^{d+1}} \models \varphi$.* □

A proof can be found, e.g., in the textbook [3].

Lemma 4.2. *For all $\ell \in \mathbb{N}$ there are* $\mathrm{FO}^2(<)$*-sentences χ_ℓ and $\chi_{\geqslant\ell}$ of size $\mathcal{O}(\ell)$ such that, for all $N \in \mathbb{N}$ $\mathcal{A}_N \models \chi_\ell$ (respectively, $\chi_{\geqslant\ell}$) iff $N = \ell$ (respectively, $N \geqslant \ell$).* □

Proof: We choose $\chi'_{\geqslant 0}(x) := (x = x)$, and, for all $\ell \geqslant 0$, $\chi'_{\geqslant \ell+1}(x) := \exists\, y\,(y < x) \wedge \chi_{\geqslant\ell}(y)$. Obviously, for all $N \in \mathbb{N}$ and all $a \in \{0,\ldots,N\}$, we have $\mathcal{A}_N \models \chi'_{\geqslant\ell}(a)$ iff $a \geqslant \ell$. Therefore, for every $\ell \in \mathbb{N}$, we can choose $\chi_{\geqslant\ell} := \exists\, x\, \chi'_{\geqslant\ell}(x)$ and $\chi_\ell := \chi_{\geqslant\ell} \wedge \neg\chi_{\geqslant\ell+1}$. ■

Theorem 4.3. *On linear orders,* FO^3($<$, *Succ*, *min*, *max*)*-sentences are $\mathcal{O}(m^4)$-succinct in* $\mathrm{FO}^2(<)$*-sentences.* □

Proof: Let ψ be an FO^3($<$, *Succ*, *min*, *max*)-sentence. Our aim is to find an $\mathrm{FO}^2(<)$-sentence χ of size $\mathcal{O}(||\psi||^4)$ such that, for all $N \in \mathbb{N}$, $\mathcal{A}_N \models \chi$ iff $\mathcal{A}_N \models \psi$.

If ψ is satisfied by *all* linear orders or by *no* linear order, χ can be chosen in a straightforward way. In all other cases we know from Lemma 4.1 that there exists a $D \in \mathbb{N}$ such that either

(1.) $\mathcal{A}_D \models \psi$ and, for all $N > D$, $\mathcal{A}_N \not\models \psi$, or

(2.) $\mathcal{A}_D \not\models \psi$ and, for all $N > D$, $\mathcal{A}_N \models \psi$.

In particular, ψ is an FO^3-sentence that distinguishes between the linear orders $\mathcal{A}_D$ and $\mathcal{A}_{D+1}$. From Corollary 3.6 we therefore know that $||\psi|| \geqslant \frac{1}{2}\sqrt{D}$.

We next construct an FO^2-sentence χ equivalent to ψ: Let χ_ℓ and $\chi_{\geqslant\ell}$ be the $\mathrm{FO}^2(<)$-sentences from Lemma 4.2. Let χ' be the disjunction of the sentences χ_ℓ for all those

$\ell \leqslant D$ with $\mathcal{A}_\ell \models \psi$. Finally, if $\mathcal{A}_{D+1} \models \psi$, then choose $\chi := \chi' \vee \chi_{\geqslant D+1}$; otherwise choose $\chi := \chi'$. Obviously, χ is an $\mathrm{FO}^2(<)$-sentence equivalent to ψ, and $||\chi|| = \mathcal{O}\big(\sum_{\ell=0}^{D+1} \ell\big) = \mathcal{O}\big(D^2\big) = \mathcal{O}(||\psi||^4)$.
This completes the proof of Theorem 4.3. ■

5. FO^3 vs. FO

Using Theorem 3.5, we will show in this section that there is an exponential succinctness gap between FO and FO^3 on linear orders.

Lemma 5.1. *For every* $\mathrm{FO}(<, \mathit{Succ}, \mathit{min}, \mathit{max})$*-sentence* φ *there is an* $\mathrm{FO}^2(<)$*-sentence* ψ *of size* $||\psi|| \leqslant 2^{\mathcal{O}(||\varphi||)}$ *which is equivalent to* φ *on the class of linear orders.* □

Proof: Let φ be an $\mathrm{FO}(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-sentence, and let d be the quantifier depth of φ. In particular, $||\varphi|| \geqslant d$.

From Lemma 4.1 we know that, for all $N \geqslant 2^{d+1}$, $\mathcal{A}_N \models \varphi$ if, and only if, $A_{2^{d+1}} \models \varphi$.

We use, for every $\ell \in \mathbb{N}$, the sentences χ_ℓ and $\chi_{\geqslant \ell}$ of Lemma 4.2. Let ψ' be the disjunction of the sentences χ_ℓ for all $\ell < 2^{d+1}$ such that $\mathcal{A}_\ell \models \varphi$. Finally, if $\mathcal{A}_{2^{d+1}} \models \varphi$, then choose $\psi := \psi' \vee \chi_{\geqslant 2^{d+1}}$; otherwise choose $\psi := \psi'$. Obviously, ψ is an $\mathrm{FO}^2(<)$-sentence equivalent to φ, and $||\psi|| = \mathcal{O}\big(\sum_{\ell=0}^{2^{d+1}} \ell\big) = \mathcal{O}\big(2^{2(d+1)}\big) = 2^{\mathcal{O}(||\varphi||)}$. ■

Lemma 5.2. *For all* $m \in \mathbb{N}$ *there are* $\mathrm{FO}^4(<)$*-sentences* φ_m *and sets* A_m *and* B_m *of interpretations, such that* $A_m \models \varphi_m$, $B_m \models \neg\varphi_m$, $||\varphi_m|| = \mathcal{O}(m)$, *and every* $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$*-sentence* ψ_m *equivalent to* φ_m *has size* $||\psi_m|| \geqslant 2^{\frac{1}{2}m-1}$. □

Proof: For every $N \in \mathbb{N}$ let $\alpha_N : \{x, y, z\} \to \{0, \ldots, N\}$ be the assignment with $\alpha_N(x) = 0$ and $\alpha_N(y) = \alpha_N(z) = N$.

For every $m \in \mathbb{N}$ we choose $A_m := \{(\mathcal{A}_{2^m}, \alpha_{2^m})\}$ and $B_m := \{(\mathcal{A}_{2^m+1}, \alpha_{2^m+1})\}$.

Step 1: Choice of φ_m*.*
Inductively we define $\mathrm{FO}^4(<)$-formulas $\varphi'_m(x, y)$ expressing that $|\mathit{diff}(x, y)| = 2^m$ via

$$\varphi'_m(x, y) \;\; := \;\; \exists z\, \forall u \;\, (u = x \vee u = y) \to \varphi'_{m-1}(z, u)$$

(and $\varphi'_0(x, y)$ chosen appropriately).
It is straightforward to see that $||\varphi'_m|| = \mathcal{O}(m)$ and that $\varphi'_m(x, y)$ expresses that $|\mathit{diff}(x, y)| = 2^m$. Therefore, $\varphi_m := \exists x\, \exists y\; \varphi'_m(x, y) \wedge \neg\exists z\, (z{<}x \vee y{<}z)$ is an $\mathrm{FO}^4(<)$-sentence with the desired properties.

Step 2: Size of equivalent FO^3*-sentences.*
For every $m \in \mathbb{N}$ let ψ_m be an $\mathrm{FO}^3(<, \mathit{Succ}, \mathit{min}, \mathit{max})$-sentence with $A_m \models \psi_m$ and $B_m \models \neg\psi_m$. From Corollary 3.6 we conclude that $||\psi_m|| \geqslant \frac{1}{2}\sqrt{2^m} = 2^{\frac{1}{2}m-1}$.
This completes the proof of Lemma 5.2. ■

From Lemma 5.1 and 5.2 we directly obtain

Theorem 5.3. *On the class of linear orders,* $\mathrm{FO}(<)$*-sentences are* $2^{\mathcal{O}(m)}$*-succinct in* $\mathrm{FO}^3(<)$*-sentences, but already* $\mathrm{FO}^4(<)$*-sentences are not* $2^{o(m)}$*-succinct in* $\mathrm{FO}^3(<)$*-sentences.* □

Note that the relation *Succ* and the constants *min* and *max* are easily definable in $\mathrm{FO}^3(<)$ and could therefore be added in Theorem 5.3.

6. FO vs. MSO

In this section we compare the succinctness of FO and the FO-expressible fragment of monadic second-order logic (for short: MSO). We write $\mathrm{Mon}\Sigma^1_1$ for the class of all MSO-formulas that consist of a prefix of existential set quantifiers, followed by a first-order formula. By $\exists X$ FO we denote the fragment of $\mathrm{Mon}\Sigma^1_1$ with only a single existential set quantifier.

Let *Tower* : $\mathbb{N} \to \mathbb{N}$ be the function which maps every $h \in \mathbb{N}$ to the tower of 2s of height h. I.e., $\mathit{Tower}(0) = 1$ and, for every $h \in \mathbb{N}$, $\mathit{Tower}(h{+}1) = 2^{\mathit{Tower}(h)}$.

We use the following notations of [5]:
For $h \geqslant 1$ let $\Sigma_h := \{0, 1,$ `<1>`, `</1>`, $\ldots,$ `<h>`, `</h>`$\}$. The "tags" `<i>` and `</i>` represent single letters of the alphabet and are just chosen to improve readability. For every $n \geqslant 1$ let $L(n)$ be the length of the binary representation of the number $n{-}1$, i.e., $L(0) = 0$, $L(1) = 1$, and $L(n) = \lfloor \log(n{-}1) \rfloor + 1$, for all $n \geqslant 2$. By $\mathrm{bit}(i, n)$ we denote the i-th bit of the binary representation of n, i.e., $\mathrm{bit}(i, n)$ is 1 if $\lfloor \frac{n}{2^i} \rfloor$ is odd, and $\mathrm{bit}(i, n)$ is 0 otherwise.
We encode every number $n \in \mathbb{N}$ by a string $\mu_h(n)$ over the alphabet Σ_h, where $\mu_h(n)$ is inductively defined as follows: $\mu_1(0) :=$ `<1></1>`, and $\mu_1(n) :=$

`<1>` $\mathrm{bit}(0, n{-}1)\, \mathrm{bit}(1, n{-}1) \cdots \mathrm{bit}(L(n){-}1, n{-}1)$ `</1>`,

for $n \geqslant 1$. For $h \geqslant 2$ we let $\mu_h(0) :=$ `<h></h>` and

$\mu_h(n) \;:=\;$ `<h>` $\mu_{h-1}(0)\, \mathrm{bit}(0, n{-}1)$
$\mu_{h-1}(1)\, \mathrm{bit}(1, n{-}1)$
$\vdots$
$\mu_{h-1}(L(n){-}1)\, \mathrm{bit}(L(n){-}1, n{-}1)$ `</h>`,

for $n \geqslant 1$. Here empty spaces and line breaks are just used to improve readability.

For $h \in \mathbb{N}$ let $H := \mathit{Tower}(h)$. Let $\Sigma^\bullet_h := \Sigma_{h+1} \cup \{\bullet\}$, and let v_h denote the following string

`<h+1>` $\mu_h(0) \bullet \;\; \mu_h(1) \bullet \cdots \mu_h(H{-}1) \bullet$ `</h+1>`.

We consider the string-language $(v_h)^+$, containing all strings that are the concatenation of one or more copies of v_h. Let w_h be the (unique) string in $(v_h)^+$ that consists of exactly 2^H copies of v_h.

We write τ_h for the signature that consists of the symbol $<$ and a unary relation symbol P_σ, for every $\sigma \in \Sigma^\bullet_h$. Nonempty strings over $\Sigma^\bullet_h$ are represented by τ_h-structures in the usual way (cf., e.g., [3]).

Lemma 6.1. *For every* $h \in \mathbb{N}$ *there is an* $\exists X\, \mathrm{FO}(\tau_h)$*-sentence* Φ_h *of size* $\mathcal{O}(h^2)$*, such that the following is true for all strings* w *over the alphabet* $\Sigma_h^{\bullet}$*:* $w \models \Phi_h$ *iff* $w = w_h$. □

The proof is given in Appendix B.

As a consequence of Lemma 6.1 and Lemma 4.1 we obtain

Theorem 6.2. *The* $\mathrm{FO}(<)$*-expressible fragment of* $\mathrm{Mon}\Sigma_1^1$ *is not Tower*$\big(o(\sqrt{m})\big)$*-succinct in* $\mathrm{FO}(<)$ *on the class of linear orders.* □

The proof is given in Appendix B.

Let us remark that, by modifying the proof of the above result, one can also show that the $\mathrm{FO}(<)$-expressible fragment of *monadic least fixed point logic*, MLFP, is non-elementarily more succinct than $\mathrm{FO}(<)$ on the class of linear orders.

7. Conclusion

Our main technical result is a lower bound on the size of a 3-variable formula defining a linear order of a given size. We introduced a new technique based on Adler-Immerman games that might be also useful in other situations. A lot of questions remain open, let us just mention a few here:

- Is first-order logic on linear orders poly(m)-succinct in its 4-variable fragment, or is there an exponential gap?

- As a next step, it would be interesting to study the succinctness of the finite-variable fragments on strings, that is, linear orders with additional unary relation symbols. It is known that on finite strings, the 3-variable fragment of first-order logic has the same expressive power as full first-order logic. Our results show that there is an at least exponential succinctness gap between the 3-variable and the 4-variable fragment. We do not know, however, if this gap is only singly exponential or larger, and we also do not know what happens beyond 4 variables.

- Another interesting task is to study the succinctness of various extensions of (finite variable fragments of) first-order logic by transitive closure operators.

- It also remains to be investigated if our results can possibly help to settle the long standing open problem of whether the 3-variable and 4-variable fragments of first-order logic have the same expressive power on the class of all ordered finite structures.

Finally, let us express our hope that techniques for proving lower bounds on succinctness will further improve in the future so that simple results such as ours will have simple proofs!

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Appendix

A. Proofs omitted in Section 3

Proof of Lemma 3.2:
We need the following notation: For a number $d \in \mathbb{N}$ and an interpretation $(\mathcal{A}, \alpha)$ choose

$$\mathit{Ord}(\mathcal{A}, \alpha) \ : \ \{0, 1, 2, 3, 4\} \to \{\mathit{min}, x, y, z, \mathit{max}\}$$

such that $\alpha\big(\mathit{Ord}(\mathcal{A}, \alpha)(i)\big) \leqslant \alpha\big(\mathit{Ord}(\mathcal{A}, \alpha)(i + 1)\big)$, for all $0 \leqslant i < 4$. Furthermore, choose

$$\mathit{Dist}_d(\mathcal{A}, \alpha) \ : \ \{ \, (i, i+1) \ : \ 0 \leqslant i < 4\} \to \{0, \ldots, 2^{d+1}\}$$

such that the following is true for all $0 \leqslant i < 4$:

$$\mathit{Dist}_d(\mathcal{A}, \alpha)(i, i+1) = \mathit{diff}\Big(\alpha\big(\mathit{Ord}(\mathcal{A}, \alpha)(i{+}1)\big), \alpha\big(\mathit{Ord}(\mathcal{A}, \alpha)(i)\big)\Big), \quad \text{or}$$

$$\mathit{Dist}_d(\mathcal{A}, \alpha)(i, i+1) = 2^{d+1} \leqslant \mathit{diff}\Big(\alpha\big(\mathit{Ord}(\mathcal{A}, \alpha)(i{+}1)\big), \alpha\big(\mathit{Ord}(\mathcal{A}, \alpha)(i)\big)\Big).$$

Finally, we define the *d-type* of $(\mathcal{A}, \alpha)$ as

$$\mathit{Type}_d(\mathcal{A}, \alpha) \ := \ \big(\, \mathit{Ord}(\mathcal{A}, \alpha), \ \mathit{Dist}_d(\mathcal{A}, \alpha)\,\big).$$

Using an Ehrenfeucht-Fraïssé game, it is an easy exercise to show the following (cf., e.g., [3]):

Lemma A.1. *Let $d \in \mathbb{N}$ and let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ be interpretations. If $\mathit{Type}_d(\mathcal{A}, \alpha) = \mathit{Type}_d(\mathcal{B}, \beta)$, then $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \beta)$ cannot be distinguished by* FO($<$)*-formulas of quantifier depth $\leqslant d$.* □

Let d be the quantifier depth of the formula ψ. We define δ to be the potential separator with $\delta(p) := 2^{d+1}$, for all $p \in \mathcal{P}_2(\{\mathit{min}, \mathit{max}, x, y, z\})$.
To show that δ is, in fact, a *separator* for $\langle A, B\rangle$, let $(\mathcal{A}, \alpha) \in A$ and $(\mathcal{B}, \beta) \in B$. Since $(\mathcal{A}, \alpha) \models \psi$ and $(\mathcal{B}, \beta) \not\models \psi$, we obtain from Lemma A.1 that $\mathit{Type}_d((\mathcal{A}, \alpha)) \neq \mathit{Type}_d((\mathcal{B}, \beta))$, i.e.,

1. $\mathit{Ord}(\mathcal{A}, \alpha) \neq \mathit{Ord}(\mathcal{B}, \beta)$, or
2. $\mathit{Dist}_d(\mathcal{A}, \alpha) \neq \mathit{Dist}_d(\mathcal{B}, \beta)$.

Therefore, there exist $u, u' \in \{\mathit{min}, \mathit{max}, x, y, z\}$ with $u \neq u'$, such that

1. $<\text{-}\mathit{type}\big(\alpha(u), \alpha(u')\big) \ \neq \ <\text{-}\mathit{type}\big(\beta(u), \beta(u')\big)$, or
2. $\mathit{diff}\big(\alpha(u), \alpha(u')\big) \neq \mathit{diff}\big(\beta(u), \beta(u')\big)$ and $\delta\big(\{u, u'\}\big) = 2^{d+1} \geqslant \mathit{MIN}\,\big\{|\mathit{diff}\big(\alpha(u), \alpha(u')\big)|\,,\ |\mathit{diff}\big(\beta(u), \beta(u')\big)|\big\}$.

Consequently, δ is a separator for $\langle A, B\rangle$, and the proof of Lemma 3.2 is complete. ■

Proof of Lemma 3.9:
By induction on the size of $\mathcal{T}$.

If $\mathcal{T}$ consists of a single node v, then $||\mathcal{T}|| = 1 \geqslant \frac{1}{2} \cdot 1$; and $1 \geqslant w(v)$, since v is a leaf.

If $\mathcal{T}$ consists of a root node v whose first child v_1 is the root of a tree $\mathcal{T}_1$ and whose second child v_2 is the root of a tree $\mathcal{T}_2$, then $||\mathcal{T}|| = 1 + ||\mathcal{T}_1|| + ||\mathcal{T}_2||$. By induction we know for $i \in \{1, 2\}$ that $||\mathcal{T}_i|| \geqslant \frac{1}{2} w(v_i)$. From the assumption we have that $w(v) \leqslant w(v_1) + w(v_2)$. Therfore, $||\mathcal{T}|| \geqslant 1 + \frac{1}{2} w(v_1) + \frac{1}{2} w(v_2) \geqslant \frac{1}{2} w(v)$.

If $\mathcal{T}$ consists of a root node v whose unique child v_1 is the root of a tree $\mathcal{T}_1$, then $||\mathcal{T}|| = 1 + ||\mathcal{T}_1||$. By induction we know that $||\mathcal{T}_1|| \geqslant \frac{1}{2} w(v_1)$. From the assumption we have that $w(v) \leqslant w(v_1) + 2$, i.e., $\frac{1}{2} w(v) \leqslant \frac{1}{2} w(v_1) + 1$. Therefore, $||\mathcal{T}|| \geqslant 1 + \frac{1}{2} w(v_1) \geqslant \frac{1}{2} w(v)$.
This completes the proof of Lemma 3.9. ■

Proof of part *(a)* of Lemma 3.8:
Let v be a *leaf* of $\mathcal{T}$ and let δ be a minimal separator for $\mathit{il}(v) =: \langle A_v, B_v\rangle$. Our aim is to show that $w(\delta) \leqslant 1$.

By Definition 3.7 we know that $\mathit{sl}(v)$ is an *atomic* formula of the form $R(u, u')$ for $R \in \{<, =, \mathit{Succ}\}$ and $u, u' \in \{\mathit{min}, \mathit{max}, x, y, z\}$. Furthermore, $A_v \models R(u, u')$ and $B_v \models \neg R(u, u')$. I.e., for all $(\mathcal{A}, \alpha) \in A_v$ and $(\mathcal{B}, \beta) \in B_v$, $<\text{-}\mathit{type}\big(\alpha(u), \alpha(u')\big) \neq <\text{-}\mathit{type}\big(\beta(u), \beta(u')\big)$, or $|\mathit{diff}\big(\alpha(u), \alpha(u')\big)| = 1 \neq |\mathit{diff}\big(\beta(u), \beta(u')\big)|$. In case that $u \neq u'$ we can define a *separator* $\tilde{\delta}$ for $\langle A_v, B_v\rangle$ via $\tilde{\delta}(p) := 1$ if $p = \{u, u'\}$ and $\tilde{\delta}(p) := 0$ otherwise. Since δ is a *minimal* separator, we obtain that $w(\delta) \leqslant w(\tilde{\delta}) = 1$.

It remains to consider the case where $u = u'$. Here, $A_v \models R(u, u)$ and $B_v \models \neg R(u, u)$. Since $R \in \{<, =, \mathit{Succ}\}$ this implies that $A_v = \emptyset$ or $B_v = \emptyset$. Therefore, according to Definition 3.1, the mapping $\tilde{\delta}$ with $\tilde{\delta}(p) = 0$, for all $p \in \mathcal{P}_2(\{\mathit{min}, \mathit{max}, x, y, z\})$, is a separator for $\langle A_v, B_v\rangle$. Hence, $w(\delta) \leqslant w(\tilde{\delta}) = 0$.
This completes the proof of part *(a)* of Lemma 3.8. ■

Proof of Lemma 3.10:
Let $\langle A, B\rangle := \mathit{il}(v)$. We need to show that $\tilde{\delta}$ is a separator for $\langle \mathcal{I}, \mathcal{J}\rangle$, for all $\mathcal{I} \in A$ and $\mathcal{J} \in B$. Let therefore $\mathcal{I} := (\mathcal{A}, \alpha) \in A$ and $\mathcal{J} := (\mathcal{B}, \beta) \in B$ be fixed for the remainder of this proof.

Since v has 2 children, we know from Definition 3.7 that $\mathit{sl}(v) = \vee$ or $\mathit{sl}(v) = \wedge$. Let us first consider the case where $\mathit{sl}(v) = \vee$.

From Definition 3.7 we know that, for $i \in \{1, 2\}$, $\mathit{il}(v_i) = \langle A_i, B\rangle$, where $A_1 \cup A_2 = A$. Therefore, there is an $i \in \{1, 2\}$ such that $\mathcal{I} \in A_i$. From the assumption we know that δ_i is a separator for $\langle A_i, B\rangle$. Therefore, there are $u, u' \in \{\mathit{min}, \mathit{max}, x, y, z\}$ with $u \neq u'$, such that $\delta_i\big(\{u, u'\}\big) \geqslant 1$ and

1. $<\text{-}\mathit{type}\big(\alpha(u), \alpha(u')\big) \ \neq \ <\text{-}\mathit{type}\big(\beta(u), \beta(u')\big)$ or

2. $\delta(\{u,u'\}) \geqslant \mathit{MIN}\,\{|\mathit{diff}(\alpha(u),\alpha(u'))|\,,\,|\mathit{diff}(\beta(u),\beta(u'))|\}$ and $\mathit{diff}(\alpha(u),\alpha(u')) \neq \mathit{diff}(\beta(u),\beta(u'))$.

Since $\tilde{\delta}(\{u,u'\}) = \delta_1(\{u,u'\})+\delta_2(\{u,u'\})$, we know that $\tilde{\delta}(\{u,u'\}) \geqslant \delta_i(\{u,u'\})$. Therefore, $\tilde{\delta}$ is a separator for $\langle \mathcal{I},\mathcal{J}\rangle$. This completes the proof of Lemma 3.10 for the case that $\mathit{sl}(v) = \vee$.

The case $\mathit{sl}(v) = \wedge$ follows by symmetry. ■

Proof of part *(b)* of Lemma 3.8:
Let v be a node of $\mathcal{T}$ that has two children v_1 and v_2. Let δ, δ_1, and δ_2, respectively, be minimal separators for $\mathit{il}(v)$, $\mathit{il}(v_1)$, and $\mathit{il}(v_2)$, respectively. Our aim is to show that $w(\delta) \leqslant w(\delta_1) + w(\delta_2)$.

Let $\tilde{\delta}$ be the *separator for* $\mathit{il}(v)$ obtained in Lemma 3.10. Since δ is a *minimal* separator for $\mathit{il}(v)$, it suffices to show that $w(\tilde{\delta}) \leqslant w(\delta_1) + w(\delta_2)$.

Using Definition 3.3, it is straightforward to check that $b(\tilde{\delta}) \leqslant b(\delta_1) + b(\delta_2)$ and $c(\tilde{\delta}) \leqslant c(\delta_1) + c(\delta_2)$. From this we obtain that

$$\begin{aligned} w(\tilde{\delta})^2 &= c(\tilde{\delta})^2 + b(\tilde{\delta}) \\ &\leqslant \big(c(\delta_1)+c(\delta_2)\big)^2 + b(\delta_1) + b(\delta_2) \\ &= c(\delta_1)^2 + b(\delta_1) + c(\delta_2)^2 + b(\delta_2) + 2c(\delta_1)c(\delta_2) \\ &\leqslant w(\delta_1)^2 + w(\delta_2)^2 + 2w(\delta_1)w(\delta_2) \\ &= \big(w(\delta_1)+w(\delta_2)\big)^2 . \end{aligned}$$

I.e., we have shown that $w(\tilde{\delta}) \leqslant w(\delta_1) + w(\delta_2)$.
This completes the proof of part *(b)* of Lemma 3.8. ■

B. Proofs omitted in Section 6

For proving Lemma 6.1 we need the following:

Lemma B.1 ([5, Lemma 8]). *For all $h \in \mathbb{N}$ there are* FO(τ_h)*-formulas* $\mathit{equal}_h(x,y)$ *of size*[1] $\mathcal{O}(h)$ *such that the following is true for all strings w over alphabet Σ_h, for all positions a,b in w, and for all numbers $m,n \in \{0,..,\mathit{Tower}(h)\}$: If a is the first position of a substring u of w that is isomorphic to $\mu_h(m)$ and if b is the first position of a substring v of w that is isomorphic to $\mu_h(n)$, then $w \models \mathit{equal}_h(a,b)$ if, and only if, $m = n$.* □

Using the above lemma, it is an easy exercise to show

Lemma B.2. *For every $h \in \mathbb{N}$ there is an* FO(τ_h)*-formula $\mathit{inc}_h(x,y)$ of size $\mathcal{O}(h)$ such that the following is true for all strings w over alphabet Σ_h, for all positions a,b in w, and for all numbers $m,n \in \{0,..,\mathit{Tower}(h)\}$: If a is the first position of a substring u of w that is isomorphic to $\mu_h(m)$ and if b is the first position of a substring v of w that is isomorphic to $\mu_h(n)$, then $w \models \mathit{inc}_h(a,b)$ if, and only if, $m{+}1 = n$.* □

[1] In [5], an additional factor $\lg h$ occurs because there a logarithmic cost measure is used for the formula size, whereas here we use a uniform measure.

We also need:

Lemma B.3. *For every $h \in \mathbb{N}$, the language $(v_h)^+$ is definable by an* FO(τ_h)*-sentence $\varphi_{(v_h)^+}$ of size $\mathcal{O}(h^2)$. I.e., for all strings w over the alphabet $\Sigma_h^\bullet$ we have $w \models \varphi_{(v_h)^+}$ if, and only if, $w \in (v_h)^+$.* □

Proof: The proof proceeds in 2 steps:

<u>*Step 1:*</u> Given $j \geqslant 1$, we say that a string w over $\Sigma_h^\bullet$ satisfies the condition $C(j)$ if, and only if, for every position x (respectively, y) in w that carries the letter `<j>` (respectively, `</j>`) the following is true: There is a position y to the right of x that carries the letter `</j>` (respectively, a position x to the left of y that carries the letter `<j>`), such that the substring u of w that starts at position x and ends at position y is of the form $\mu_j(n)$ for some $n \in \{0,..,\mathit{Tower}(j){-}1\}$.

We will construct, for all $j \in \{1,..,h\}$, FO(τ_h)-sentences ok_j of size $\mathcal{O}(j)$ such that the following is true for all $j \leqslant h$ and all strings w over $\Sigma_h^\bullet$ that satisfy the conditions $C(j')$ for all $j' < j$:

$$w \models \mathit{ok}_j \quad \text{iff} \quad w \text{ satisfies the condition } C(j).$$

Simultaneously we will construct, for all $j \in \{1,..,h\}$, FO(τ_h)-sentences $\mathit{max}_j(x)$ of size $\mathcal{O}(j)$ such that the following is true for all $j \leqslant h$, all strings w that satisfy the conditions $C(1),..,C(j)$, and all positions x in w:

$$w \models \mathit{max}_j(x) \quad \text{iff} \quad x \text{ is the starting position of a substring of } w \text{ of the form } \mu_j(\mathit{Tower}(j){-}1).$$

For the base case $j = 1$ note that $\mathit{Tower}(1){-}1 = 1$ and, by the definition of $\mu_1(n)$, $\mu_1(0) =$ `<1></1>` and $\mu_1(1) =$ `<1>0</1>`. It is straightforward to write down a formula ok_1 that expresses the condition $C(1)$. Furthermore, $\mathit{max}_1(x)$ states that the substring of length 3 starting at position x is of the form `<1>0</1>`.

For $j > 1$ assume that the formula max_{j-1} has already been constructed. For the construction of the formula ok_j we assume that the underlying string w satisfies the conditions $C(1),..,C(j{-}1)$.

The formula ok_j states that whenever x (respectively, y) is a position in w that carries the letter `<j>` (respectively, `</j>`) the following is true: There is a position y to the right of x that carries the letter `</j>` (respectively, a position x to the left of y that carries the letter `<j>`), such that the substring u of w that starts at position x and ends at position y is of the form $\mu_j(n)$ for some $n \in \{0,..,\mathit{Tower}(j){-}1\}$, i.e.,

1. the letters `<j>` and `</j>` only occur at the first and the last position of u,

2. whenever a position x' carries the letter `</j-1>`, position $x'+1$ carries the letter 0 or 1, and position $x'+2$ carries the letter `<j-1>` or `</j>`,

3. either $u =$ `<j></j>`, or the prefix of length 3 of u is of the form `<j><j-1></j-1>`,

4. whenever x' and y' are positions in u carrying the letter `<j-1>` such that $x' < y'$ and no position between x' and y' carries the letter `<j-1>`, the formula $\mathit{inc}_{j-1}(x', y')$ from Lemma B.2 is satisfied,

5. if the rightmost position x'' in u that carries the letter `<j-1>` satisfies the formula $\mathit{max}_{j-1}(x'')$, then there must be at least one position x''' in u that carries the letter 0 such that $x'''-1$ carries the letter `</j-1>`.

Note that items 1.–4. ensure that u is indeed of the form $\mu_j(n)$, for some $n \in \mathbb{N}$. Item 5 guarantees that $n \in \{0, .., \mathit{Tower}(j)-1\}$ because of the following: recall from the definition of the string $\mu_j(n)$ that $\mu_j(n)$ involves the (reverse) binary representation of the number $n-1$. In particular, for $n := \mathit{Tower}(j)-1$, we need the (reverse) binary representation of the number $\mathit{Tower}(j)-2$, which is of the form $011\cdots11$ and of length $\mathit{Tower}(j-1)$, i.e., its highest bit has the number $\mathit{Tower}(j-1)-1$.

It is straightforward to see that the items 1.–5. and therefore also the formula ok_j can be formalized by an FO(τ_h)-formula of size $\mathcal{O}(j)$, and that this formula exactly expresses condition $C(j)$.

Furthermore, the formula $\mathit{max}_j(x)$ assumes that x is the starting position of a substring u of w of the form $\mu_j(n)$, for some $n \in \mathbb{N}$; and $\mathit{max}_j(x)$ states that

1. the (reverse) binary representation of n, i.e., the $\{0,1\}$-string built from the letters in u that occur directly to the right of letters `</j-1>`, is of the form $011\cdots11$, and

2. the highest bit of n has the number $\mathit{Tower}(j-1)-1$, i.e., the rightmost position y in u that carries the letter `<j-1>` satisfies the formula $\max_{j-1}(y)$.

Obviously, $\mathit{max}_j(x)$ can be formalized in FO(τ_h) by a formula of size $\mathcal{O}(j)$. Finally, this completes Step 1.

Step 2: A string w over $\Sigma_h^\bullet$ belongs to the language $(v_h)^+$ if, and only if,

- w satisfies $\mathit{ok}_1 \wedge \cdots \wedge \mathit{ok}_h$,

- the first position in w carries the letter `<h+1>`, the last position in w carries the letter `</h+1>`, the letter `<h+1>` occurs at a position $x > 1$ iff position $x-1$ carries the letter `</h+1>`, and the letter $\bullet$ occurs at a position x iff position $x-1$ carries the letter `</h>` and position $x+1$ carries the letter `<h>` or `</h+1>`,

- whenever x (respectively, y) is a position in w that carries the letter `<h+1>` (respectively, `</h+1>`) the following is true: There is a position y to the right of x that carries the letter `</h+1>` (respectively, a position x to the left of y that carries the letter `<h+1>`), such that the substring u of w that starts at position x and ends at position y is of the form v_h, i.e.,

  ⋆ the letters `<h+1>` and `</h+1>` only occur at the first and the last position of u,

  ⋆ the prefix of length 3 of u is of the form `<h+1><h></h>`, and the suffix of length 3 of u is of the form `</h>` $\bullet$ `</h+1>`,

  ⋆ whenever x' and y' are positions in u carrying the letter `<h>` such that $x' < y'$ and no position between x' and y' carries the letter `<h>`, the formula $\mathit{inc}_h(x', y')$ from Lemma B.2 is satisfied,

  ⋆ the rightmost position x'' in u that carries the letter `<h>` satisfies the formula $\mathit{max}_h(x'')$.

Using the formulas constructed in Step 1 and the preceding lemmas, it is straightforward to see that this can be formalized by an FO(τ_h)-formula $\varphi_{(v_h)^+}$ of size $\mathcal{O}(h^2)$. This finally completes the proof of Lemma B.3. ■

Proof of Lemma 6.1:
To determine whether an input string w is indeed the string w_h, one can proceed as follows: First, we make sure that the underlying string w belongs to $(v_h)^+$ via the FO(τ_h)-formula $\varphi_{(v_h)^+}$ of Lemma B.3. Afterwards we, in particular, know that in each `<h+1>`$\cdots$`</h+1>`-block is of the form v_h and therefore contains exactly H positions that carry the letter $\bullet$. Now, to each $\bullet$-position in w we assign a letter from $\{0,1\}$ in such a way that the $\{0,1\}$-string built from these assignments is an H-*numbering*, i.e., of the form

- $\mathrm{BIN}_H(0)\,\mathrm{BIN}_H(1)\,\mathrm{BIN}_H(2)\cdots\mathrm{BIN}_H(n)$,
  for some $n < 2^H$, or

- $\mathrm{BIN}_H(0)\,\mathrm{BIN}_H(1)\cdots\mathrm{BIN}_H(2^H-1)\,\Big(\mathrm{BIN}_H(0)^m\Big)$,
  for some $m \geqslant 0$.

Here, $\mathrm{BIN}_H(n)$ denotes the reverse binary representation of length H of the number $n < 2^H$. For example, $\mathrm{BIN}_4(2) = 0100$ and $\mathrm{BIN}_4(5) = 1010$. Of course, w is the string w_h, i.e., consists of exactly 2^H copies of v_h, if and only if the H-numbering's assignments in the rightmost copy of v_h form the string $\mathrm{BIN}_H(2^H-1)$, i.e., if and only if every $\bullet$-position in this copy of v_h was assigned the letter 1.
One way of assigning letters from $\{0,1\}$ to the $\bullet$-positions in w is by choosing a set X of $\bullet$-positions with the intended meaning that a $\bullet$-position x is assigned the letter 1 if $x \in X$ and the letter 0 if $x \notin X$.

Using the FO(τ_h)-formulas $equal_h$ of Lemma B.1 and $\varphi_{(v_h)^+}$ of Lemma B.3, it is straightforward to construct the desired $\exists X$ FO(τ_h)-formula Φ_h of size $\mathcal{O}(h)$.
This completes the proof of Lemma 6.1. ■

Proof of Theorem 6.2:
Recall that, for every $N \in \mathbb{N}$, $\mathcal{A}_N$ denotes the linear order with universe $\{0, .., N\}$.

For every $h \in \mathbb{N}$ let $\ell(h) := |w_h| - 1$, where $|w_h|$ denotes the length of the string w_h. We say that a sentence χ *defines* the linear order $\mathcal{A}_{\ell(h)}$ if, and only if, $\mathcal{A}_{\ell(h)}$ is the unique structure in $\{\mathcal{A}_N : N \in \mathbb{N}\}$ that satisfies χ. For every $h \in \mathbb{N}$ we show the following:

(a) Every FO($<, Succ, min, max$)-sentence ψ_h that defines $\mathcal{A}_{\ell(h)}$ has size $||\psi_h|| \geqslant Tower(h)$.

(b) There is a Mon$\Sigma^1_1(<)$-sentence Ψ_h of size $||\Psi_h|| = \mathcal{O}(h^2)$ that defines $\mathcal{A}_{\ell(h)}$.

Ad (a):
Since w_h consists of $2^{Tower(h)}$ copies of v_h, we know that $\ell(h) \geqslant 2^{Tower(h)}$. Therefore, every FO($<$)-sentence ψ_h that defines $\mathcal{A}_{\ell(h)}$ has quantifier depth, and therefore size, at least $Tower(h)$ (cf., Lemma 4.1).

Ad (b):
Let $\exists X\, \varphi_h$ be the $\exists X$ FO(τ_h)-sentence obtained from Lemma 6.1. It is straightforward to formulate an FO(τ_h)-sentence ξ_h of size $\mathcal{O}(h^2)$ which expresses that every element in the underlying structure's universe belongs to exactly one of the sets P_σ, for $\sigma \in \Sigma_h^\bullet$.
The Mon$\Sigma^1_1(<)$-sentence

$$\Psi_h \quad := \quad \big(\exists P_\sigma\big)_{\sigma \in \Sigma_h^\bullet} \ \exists X \quad (\, \xi_h \wedge \varphi_h \,)$$

expresses that the nodes of the underlying linear order can be labelled with letters in $\Sigma_h^\bullet$ in such a way that one obtains the string w_h. This is possible if, and only if, the linear order has length $|w_h|$. I.e., Ψ_h *defines* $\mathcal{A}_{\ell(h)}$. Furthermore, $||\Psi_h|| = \mathcal{O}(h^2)$, because $||\xi_h|| = \mathcal{O}(h^2)$ and $||\varphi_h|| = \mathcal{O}(h^2)$.
This completes the proof of Theorem 6.2 ■