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Abstract. In 1983, Chvátal, Trotter and the two senior authors proved that for any Δ there exists B so that, for any n, any 2-coloring of the edges of the complete graph K_N with $N \geq Bn$ vertices yields a monochromatic copy of any graph H that has n vertices and maximum degree Δ. We prove that the complete graph may be replaced by a sparser graph G that has N vertices and $O(N^{2-1/\Delta} \log^{1/\Delta} N)$ edges, with $N = \lfloor B'n \rfloor$ for some constant B' that depends only on Δ. Consequently, the so called *size-Ramsey number* of any H with n vertices and maximum degree Δ is $O(n^{2-1/\Delta} \log^{1/\Delta} n)$. Our approach is based on random graphs; in fact, we show that the classical Erdős–Rényi random graph with the numerical parameters above satisfies a stronger partition property with high probability, namely, that any 2-coloring of its edges contains a monochromatic *universal graph* for the class of graphs on n vertices and maximum degree Δ.

The main tool in our proof is the regularity method, adapted to a suitable sparse setting. The novel ingredient developed here is an embedding strategy that allows one to embed bounded degree graphs of linear order in certain pseudorandom graphs. Crucial to our proof is a rather surprising phenomenon, namely, the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed.

1. Introduction and results

The regularity method has proved to be a powerful tool in asymptotic combinatorics. Regular decompositions of graphs and hypergraphs reveal much of the structure of such objects, and have been fundamental in approaching diverse problems in the area (see [26, 29]). The regularity method for *dense graphs* is the best developed direction in this line of research, with a long history of applications and such surprising tools as the blow-up lemma [27, 28]. Thanks to recent advances [18, 30, 34], one is now able to apply the regularity method to *hypergraphs*; for instance, one may now give a fully combinatorial proof of theorems such as the Furstenberg–Katznelson theorem [15] on the existence of homothetic copies of finite configurations in dense subsets of the integer lattice, generalizing [35] to arbitrary dimensions (see, e.g., [31]). The regularity method for *sparse graphs* is, however, still under development: it appears that even the embedding lemma for graphs of constant size has not been proved in its full generality or strength (see,

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e.g., [17, 23, 25]). In this paper, we contribute to the development of the regularity method for sparse graphs, providing an embedding strategy for large graphs of bounded degree in the sparse setting. As an application, we prove a numerical result in Ramsey theory: we prove an upper bound for a variant of the Ramsey number for graphs of bounded degree (for numbers in Ramsey theory, see [19]).

For graphs G and H, write $G \longrightarrow H$ if G contains a monochromatic copy of H for every 2-coloring of the edges of G. Erdős, Faudree, Rousseau and Schelp [13] considered the question of how few edges G may have if $G \longrightarrow H$. Following [13] we denote the *size-Ramsey number* $\widehat{r}(H) = \min\{e(G)\colon\ G \longrightarrow H\}$, where $e(G)$ denotes the cardinality of the edge set of G.

For example $\widehat{r}(K_{1,n}) = 2n-1$ for the star $K_{1,n}$ on $n+1$ vertices. In [6] Beck disproved a conjecture of Erdős [12] and showed that $\widehat{r}(P_n) \le 900n$. More generally, it follows from the result of Friedman and Pippenger [14] that the size-Ramsey number of bounded degree trees grows linearly with the size of the tree (for further results in this direction, see [7, 20]). Moreover, it was proved in [21] that cycles also have linear size-Ramsey numbers. Beck asked in [7] if $\widehat{r}(H)$ is always linear in the number of vertices of H for graphs H of bounded degree. This was disproved by Rödl and Szemerédi [33], who proved that there are graphs of order n, maximum degree three, and $\widehat{r}(H) \ge n\log^c n$ for some constant $c > 0$. These authors also conjectured that, for every $\Delta \ge 3$, there exists $\varepsilon = \varepsilon(\Delta) > 0$ such that

$$n^{1+\varepsilon} \le \widehat{r}_{\Delta,n} := \max\{\widehat{r}(H)\colon\ H \in \mathcal{H}_{\Delta,n}\} \le n^{2-\varepsilon}\,, \tag{1}$$

where $\mathcal{H}_{\Delta,n}$ is the class of all n-vertex graphs with maximum degree at most Δ, up to isomorphism. In this paper, we prove the upper bound conjectured in (1).

In fact, our proof method yields a stronger result. Let us say that a graph is $\mathcal{H}_{\Delta,n}$-*universal* if it contains every member of $\mathcal{H}_{\Delta,n}$ as a subgraph. Furthermore, let us say that a graph is *partition universal for the class of graphs* $\mathcal{H}_{\Delta,n}$ if any 2-coloring of its edges contains a monochromatic $\mathcal{H}_{\Delta,n}$-universal graph. We shall establish for every Δ the existence of a graph G with $O(n^{2-1/\Delta}\log^{1/\Delta} n)$ edges that is partition universal for $\mathcal{H}_{\Delta,n}$.

Theorem 1. *For every $\Delta \ge 2$ there exist constants B and C such that for every n and N satisfying $N \ge Bn$ there exists a graph G on N vertices and at most $CN^{2-1/\Delta}\log^{1/\Delta} N$ edges that is is partition universal for $\mathcal{H}_{\Delta,n}$. In particular, $G \longrightarrow H$ for every $H \in \mathcal{H}_{\Delta,n}$.*

Remark 2. (*i*) As observed in [1], one can show that the number of edges in any $\mathcal{H}_{\Delta,n}$-universal graph is $\Omega(n^{2-2/\Delta})$ and, hence, the exponent $2-1/\Delta$ of N in Theorem 1 cannot be reduced to $2-2/\Delta-\varepsilon$ for any given $\varepsilon > 0$. For completeness, let us quickly see how to obtain this lower bound on the number of edges M in an $\mathcal{H}_{\Delta,n}$-universal graph G. Let us suppose first that $n\Delta$ is even. Note that we must have

$$\binom{M}{n\Delta/2} \ge \frac{1}{n!}L_{\Delta,n}, \tag{2}$$

where $L_{\Delta,n}$ denotes the number of labeled graphs on n vertices that are Δ-regular. Bender and Canfield [8] showed that, for any fixed Δ, as $n \to \infty$ with $n\Delta$ even, we have

$$L_{\Delta,n} = (1+o(1))\sqrt{2}e^{-(\Delta^2-1)/4}\left(\frac{\Delta^{\Delta/2}}{e^{\Delta/2}\Delta!}\right)^n n^{\Delta n/2}.$$

Therefore, for $n\Delta$ even, $L_{\Delta,n} = \Omega(c^n n^{n\Delta/2})$ for a constant $c = c(\Delta)$. Combining this with (2), we see that $(2\mathrm{e}M/n\Delta)^{n\Delta/2} \geq \binom{M}{n\Delta/2} \geq L_{\Delta,n}/n! = \Omega(c^n n^{n\Delta/2}/n^n)$, whence $M = \Omega(n^{2-2/\Delta})$, as required. If $n\Delta$ is odd, simply observe that an $\mathcal{H}_{\Delta,n}$-universal graph is also $\mathcal{H}_{\Delta-1,n}$-universal.

We mention that a recent, remarkable result of Alon and Capalbo [4] confirms the existence of $\mathcal{H}_{\Delta,n}$-universal graphs with $O(n^{2-2/\Delta})$ edges (see also [1, 2, 3].

(*ii*) A weaker version of Theorem 1, with $|E(G)| = N^{2-1/2\Delta+o(1)}$, was proved earlier by Kohayakawa, Rödl, and Szemerédi (unpublished).

Let $G(N, p)$ be the standard random graph on N vertices, with all the edges present with probability p, independently of one another (see [9, 22] for the theory of random graphs). To prove Theorem 1, we shall show that $G(N, p)$ with an appropriate choice of $p = p(N)$ is as required with high probability.

Theorem 3. *For every $\Delta \geq 2$ there exists a constant C for which the following holds. Let $p = p(N) = C(\log N/N)^{1/\Delta}$. Then*

$$\lim_{n\to\infty} \mathbb{P}\left(G(N,p) \text{ is partition universal for } \mathcal{H}_{\Delta,n}\right) = 1. \tag{3}$$

Remark 4. (*i*) In Theorem 1, we have restricted ourselves to the 2-color case for simplicity. One may easily prove the same result for more than two colors (the constants B and C would then depend on both Δ and on the number of colors). Similarly, Theorem 3 holds as stated for any fixed number of colors, that is, we may generalize the notion of partition universality to any fixed number of colors r and prove the same result (the constant C would then depend on both Δ and r).

(*ii*) Theorem 1 follows from Theorem 3. In the remainder of this paper, we focus our attention on the proof of Theorem 3.

The main tool in our proof of Theorem 3 is the regularity method, adapted to the appropriate sparse and random setting. The key novel ingredient in our approach is an embedding strategy that allows one to embed bounded degree graphs of linear order in suitably pseudorandom graphs (see the proof of Lemma 18). Crucial in the proof is a rather surprising phenomenon, namely, the fact that regularity is typically inherited at a scale that is much finer than the scale at which it is assumed. This phenomenon was first spelt out in full in [24], but we use an improved version proved in [16].

Organization. This paper is organized as follows. In Section 2 we recall some basic facts about regularity, including the results on inheritance of regularity proved in [16] (see Section 2.1). In Section 3.3, the results on the hereditary nature of regularity, in the form that is required here, are derived from the results quoted in Section 2.1. Other relevant results on random graphs are given in Sections 3.1 and 3.2. The proof of Theorem 3 is given in Section 4. We conclude with some remarks and open problems in Section 5.

2. The sparse regularity lemma

Let $G = (V, E)$ be a graph. Suppose $0 < p \leq 1$, $\eta > 0$ and $K > 1$. For two disjoint subsets X, Y of V, we let $e_G(X, Y)$ be the number of edges of G with one endpoint in X and the other enpoint in Y. Furthermore, we let

$$d_{G,p}(X,Y) = \frac{e_G(X,Y)}{p|X||Y|},$$

which we refer to as the *p-density* of the pair (X,Y). We say that G is an (η, K)-*bounded* graph with respect to density p if for all pairwise disjoint sets X, $Y \subseteq V$, with $|X|$, $|Y| \geq \eta|V|$, we have

$$e_G(X,Y) \leq Kp|X||Y|\,.$$

For $\varepsilon > 0$ fixed and X, $Y \subseteq V$, $X \cap Y = \emptyset$, we say that the pair (X,Y) is (ε, p)-*regular* if for all $X' \subseteq X$ and $Y' \subseteq Y$ with

$$|X'| \geq \varepsilon|X| \quad \text{and} \quad |Y'| \geq \varepsilon|Y|,$$

we have

$$|d_{G,p}(X,Y) - d_{G,p}(X',Y')| \leq \varepsilon.$$

Note that for $p = 1$ we get the well-known definition of ε-regularity [36].

Let $\dot{\bigcup}_{j=0}^{t} V_j$ be a partition of V. We call V_0 the *exceptional class*. This partition is called (ε, t)-*equitable* if $|V_0| \leq \varepsilon|V|$ and $|V_1| = \cdots = |V_t|$.

We say that an (ε, t)-equitable partition $\dot{\bigcup}_{j=0}^{t} V_j$ of V is (ε, G, p)-*regular* if all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq k$, are (ε, p)-regular. Now we state a variant of the Szemerédi's regularity lemma [36] for sparse graphs, which was observed independently by Kohayakawa and Rödl (see, e.g., [23, 25]).

Theorem 5 (Sparse regularity lemma)**.** *For any $\varepsilon > 0$, $K > 1$, and $t_0 \geq 1$, there exist constants T_0, η, and N_0 such that any graph G with at least N_0 vertices that is (η, K)-bounded with respect to density $0 < p \leq 1$ admits an (ε, t)-equitable (ε, G, p)-regular partition of its vertex set with $t_0 \leq t \leq T_0$.* □

2.1. **The hereditary nature of sparse regularity.** We shall also use the fact that ε-regularity is typically inherited on "small" (sublinear) subsets. This was essentially observed for the classical notion of (dense) regular pairs by Duke and Rödl [11] and for sparse regular pairs in [16, 24]. Here we shall use a result from [16] regarding the hereditary nature of (ε, α, p)-*denseness* (or "one sided-regularity").

Definition 6. *Let α, $\varepsilon > 0$, and $0 < p \leq 1$ be given and let $G = (V,E)$ be a graph. For sets X, $Y \subseteq V$, $X \cap Y = \emptyset$, we say that the pair (X,Y) is (ε, α, p)-dense if for all $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, we have*

$$d_{G,p}(X',Y') \geq \alpha - \varepsilon.$$

It follows immediately from the definition that (ε, α, p)-denseness is inherited on large sets, i.e., that for a (ε, α, p)-dense pair (X,Y) and any sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \mu|X|$ and $|Y'| \geq \mu|Y|$ the pair (X',Y') is $(\varepsilon/\mu, \alpha, p)$-dense. The following result from [16] states that this "denseness-property" is even inherited on randomly chosen subsets of much smaller size with overwhelming probability.

Theorem 7 ([16, Theorem 3.6])**.** *For every α, $\beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon') > 0$ and $L = L(\alpha, \varepsilon')$ such that, for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < p < 1$, every (ε, α, p)-dense pair (X,Y) in a graph G satisfies that the number of sets $X' \subseteq X$ with $|X'| = m \geq L/p$ such that (X',Y) is an $(\varepsilon', \alpha, p)$-dense pair is at least $(1-\beta^m)\binom{|X|}{m}$.* □

The following is a direct consequence of Theorem 7, which we obtain by applying it first to X and then to subsets of Y.

Corollary 8 ([16, Corollary 3.8])**.** *For every α, $\beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon') > 0$ and $L = L(\alpha, \varepsilon')$ such that for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < p < 1$, every (ε, α, p)-dense pair (X, Y) in a graph G satisfies that the number of pairs (X', Y') of sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = m_1 \geq L/p$ and $|Y'| = m_2 \geq L/p$ such that (X', Y') is an $(\varepsilon', \alpha, p)$-dense pair is at least $(1 - \beta^{\min\{m_1, m_2\}})\binom{|X|}{m_1}\binom{|Y|}{m_2}$.* □

3. Properties of the random graph

In this section we shall verify a few properties of random graphs that will be useful for the proof of Theorem 1.

3.1. Uniform edge distribution.

We start with a well known fact, which follows easily from the properties of the binomial distribution, concerning the edge distribution of $G(N, p)$.

Definition 9. *For an integer N and $0 < p \leq 1$ we define the family of graphs $\mathcal{U}_{N,p}$ on $[N] = \{1, \ldots, N\}$ with uniform edge distribution by*

$$\mathcal{U}_{N,p} := \Big\{G\colon\ V(G) = [N] \text{ and } \forall\, U, W \subseteq V(G) \text{ with } U \cap W = \emptyset,\ |U| \geq \tfrac{N}{\log N}, \\ \text{and } |W| \geq \tfrac{N}{\log N} \text{ we have } e_G(U, W) = (1 \pm \tfrac{1}{\log N})p|U||W|\Big\}.$$

The following proposition follows directly from the Chernoff bound for binomially distributed random variables.

Proposition 10. *If $p = p(N) \gg (\log N)^4/N$, then $\mathbb{P}(G(N, p) \in \mathcal{U}_{N,p}) = 1 - o(1)$.* □

In Proposition 10 and in the remainder of this paper, $o(1)$ denotes a function that tends to 0 as $N \to \infty$. We also use the symbols $\ll$ and $\gg$; e.g., we write $f(N) \ll g(N)$ to mean that $f(N)/g(N) \to 0$ as $N \to \infty$.

3.2. Expansion properties of neighbourhoods.

For a graph $G = (V, E)$ and an integer $k \geq 1$, we define the auxiliary, bipartite graph $\Gamma(k, G) = (\binom{V}{k} \dot\cup V, E_{\Gamma(k,G)})$ by

$$(K, v) \in E_{\Gamma(k,G)} \iff \{w, v\} \in E(G) \text{ for all } w \in K\,. \tag{4}$$

Proposition 12, given below, states that if G is the random graph $G(N, p)$, then the graph $\Gamma(k, G)$ has no "dense patches". More precisely, we consider the following property.

Definition 11. *Let integers N and $k \geq 1$ and reals $\xi > 0$ and $0 < p \leq 1$ be given. We say that a graph $G = (V, E)$ with $V = [N]$ has the neighbourhood expansion property $\mathcal{E}^k_{N,p}(\xi)$ if for every $U \subseteq V$ and every family $\mathcal{F}_k \subseteq \binom{V}{k}$ of pairwise disjoint k-sets with*

(i) $|\mathcal{F}_k| \leq \xi N$ and
(ii) $|U| \leq |\mathcal{F}_k|$

we have

$$e_{\Gamma(k,G)}(\mathcal{F}_k, U) \leq p^k |\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|\,. \tag{5}$$

We show that for appropriate p the random graph $G(N, p)$ asymptotically almost surely has property $\mathcal{E}^k_{N,p}(\xi)$.

Proposition 12. *For every integer $k \geq 1$ and real $\xi > 0$, there exists $C > 1$ such that if $p > C(\log N/N)^{1/k}$, then $\mathbb{P}(G(N,p) \in \mathcal{E}^k_{N,p}(\xi)) = 1 - o(1)$.*

Proof. For given k and ξ we let C be a constant satisfying

$$C^k > k/\xi\,.$$

Let $\mathcal{F}_k$ and U satisfy (i) and (ii) of Definition 11. We consider two cases depending on the size of $\mathcal{F}_k$.

Case 1 ($|\mathcal{F}_k| \geq N/\log N$). Observe that for fixed $\mathcal{F}_k$ and U the edges of $\Gamma[\mathcal{F}_k, U] = \Gamma(k, G(N,p))[\mathcal{F}_k, U]$ appear independently with probability p^k. Thus $e_\Gamma(\mathcal{F}_k, U)$ is a binomial random variable with distribution $\mathrm{Bi}(p^k, |\mathcal{F}_k||U|)$. From Chernoff's inequality

$$\mathbb{P}(X \geq \mathbb{E}X + t) \leq \exp(-t)$$

for a binomial random variable X and $t \geq 6\mathbb{E}X$ (see e.g. [22, Corollary 2.4]), we infer

$$\mathbb{P}\big(e_\Gamma(\mathcal{F}_k, U) > p^k|\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|\big) \leq \exp\big(-6\xi N p^k |\mathcal{F}_k|\big)\,,$$

since we have $|U| \leq |\mathcal{F}_k| \leq \xi N$ from (i) and (ii) of Definition 11.

Moreover, the number of choices for $\mathcal{F}_k$ (satisfying the assumptions of this case) and U is at most $\sum_{f=N/\log N}^{\xi N} N^{kf}$ and 2^N, respectively, and since

$$\sum_{f=N/\log N}^{\xi N} N^{kf} 2^N \exp(-6\xi N p^k f) \to 0$$

as $N \to \infty$ follows from the choice of $C^k > k/\xi$ and $p > C(\log N/N)^{1/k}$, the proposition is established in this case.

Case 2 ($|\mathcal{F}_k| < N/\log N$). The analysis in this case is very similar to the first, but instead of Chernoff's inequality we use that

$$\mathbb{P}(X \geq t) \leq q^t \binom{M}{t} \leq \left(\frac{\mathrm{e}qM}{t}\right)^t$$

for a binomial random variable $X \sim \mathrm{Bi}(q, M)$. Consequently,

$$\begin{aligned}\mathbb{P}\big(e_\Gamma(\mathcal{F}_k, U) \geq p^k|\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|\big) &\leq \mathbb{P}\big(e_\Gamma(\mathcal{F}_k, U) \geq 6\xi N p^k |\mathcal{F}_k|\big)\\ &\leq \left(\frac{\mathrm{e}|U|}{6\xi N}\right)^{6\xi N p^k |\mathcal{F}_k|} \leq \exp\big(-6\xi N p^k |\mathcal{F}_k| \ln(2\xi N/|U|)\big)\,.\end{aligned}$$

In this case, the number of choices for the pair $(\mathcal{F}_k, U)$ is at most $\sum_{f=1}^{N/\log N} \sum_{u=1}^{f} N^{kf}\binom{N}{u}$. Consequently, from the union bound we infer that the probability that there exists a family $\mathcal{F}_k$ and a set U with $|U| \leq |\mathcal{F}_k| < N/\log N$ such that $e_\Gamma(\mathcal{F}_k, U) \geq p^k|\mathcal{F}_k||U| + 6\xi N p^k |\mathcal{F}_k|$ is at most

$$\sum_{f=1}^{N/\log N} \sum_{u=1}^{f} \exp\big(kf\ln N + u\ln(\mathrm{e}N/u) - 6\xi N p^k f \ln(2\xi N/u)\big) \to 0\,,$$

as $N \to \infty$ since $p^k N \gg \log N/\log\log N$.

This concludes the proof of Proposition 12. □

3.3. **Hereditary nature of (ε,α,p)-denseness.** In this section we shall show that in the random graph $G(N,p)$ all sufficiently large (not necessarily induced) 3-partite subgraphs, say, with vertex set $X\dot\cup Y\dot\cup Z$, in which all the three pairs (X,Y), (X,Z) and (Y,Z) are (ε,α,p)-dense, have the following property: The (ε,α,p)-denseness of the pair (Y,Z) is "typically" inherited on the one-sided neighbourhood $(N(x)\cap Y,Z)$ as well as on the two-sided neighbourhood $(N(x)\cap Y, N(x)\cap Z)$ for $x\in X$. Below we introduce classes $\mathcal{B}_p^{\mathrm{I}}$ and $\mathcal{B}_p^{\mathrm{II}}$ of "bad" tripartite graphs, which fail to have the above one-sided and two-sided property (for similar concepts see [24]).

Definition 13. *Let integers m_1, m_2, and m_3 and reals $\alpha,\varepsilon',\varepsilon,\mu>0$, and $0<p\le 1$ be given.*

(I) *Let $\mathcal{B}_p^{\mathrm{I}}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ be the family of tripartite graphs with vertex set $X\dot\cup Y\dot\cup Z$, where $|X|=m_1$, $|Y|=m_2$, and $|Z|=m_3$, satisfying*
  - (*a*) *(X,Y) and (Y,Z) are (ε,α,p)-dense pairs and*
  - (*b*) *there exists $X'\subseteq X$ with $|X'|\ge\mu|X|$ such that $(N(x)\cap Y,Z)$ is not an (ε',α,p)-dense pair for every $x\in X'$.*

(II) *Let $\mathcal{B}_p^{\mathrm{II}}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ be the family of tripartite with vertex set $X\dot\cup Y\dot\cup Z$, where $|X|=m_1$, $|Y|=m_2$, and $|Z|=m_3$, satisfying*
  - (*a*) *(X,Y), (X,Z), and (Y,Z) are (ε,α,p)-dense pairs and*
  - (*b*) *there exists $X'\subseteq X$ with $|X'|\ge\mu|X|$ such that $(N(x)\cap Y,N(x)\cap Z)$ is not an (ε',α,p)-dense pair for every $x\in X'$.*

Definition 14. *For integers N and $\Delta\ge 2$ and reals α, γ, ε', ε, $\mu>0$ and $0<p\le 1$ we say that a graph $G=(V,E)$ with $V=[N]$ has the denseness property $\mathcal{D}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon',\varepsilon,\mu)$, if G contains no member from*

$$\mathcal{B}_p^{\mathrm{I}}(m_1^{\mathrm{I}},m_2^{\mathrm{I}},m_3^{\mathrm{I}},\alpha,\varepsilon',\varepsilon,\mu)\cup\mathcal{B}_p^{\mathrm{II}}(m_1^{\mathrm{II}},m_2^{\mathrm{II}},m_3^{\mathrm{II}},\alpha,\varepsilon',\varepsilon,\mu)$$

with $m_1^{\mathrm{I}},m_3^{\mathrm{I}}\ge\gamma p^{\Delta-1}N$ and $m_2^{\mathrm{I}},m_1^{\mathrm{II}},m_2^{\mathrm{II}},m_3^{\mathrm{II}}\ge\gamma p^{\Delta-2}N$ as a (not necessarily induced) subgraph.

Proposition 15. *For an integer $\Delta\ge 2$ and positive reals α, ε', and μ there exists*

$$\varepsilon=\varepsilon(\Delta,\alpha,\varepsilon',\mu)>0 \tag{6}$$

such that for every $\gamma>0$ there exists $C>1$ such that if $p>C(\log N/N)^{1/\Delta}$, then

$$\mathbb{P}(G(N,p)\in\mathcal{D}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon',\varepsilon,\mu))=1-o(1).$$

3.3.1. *Proof of Proposition 15.* We first verify Proposition 15 for the special case in which $m_1^{\mathrm{I}}=pm_2^{\mathrm{I}}=m_3^{\mathrm{I}}$ and $m_1^{\mathrm{II}}=m_2^{\mathrm{II}}=m_3^{\mathrm{II}}$. To that end, we consider the families of graphs $\mathcal{B}_p^{\mathrm{I}}(m,\alpha,\varepsilon',\varepsilon,\mu)$ and $\mathcal{B}_p^{\mathrm{II}}(m,\alpha,\varepsilon',\varepsilon,\mu)$ for $m\in\mathbb{N}$ and α, ε', ε, $\mu>0$ defined as

$$\mathcal{B}_p^{\mathrm{I}}(m,\alpha,\varepsilon',\varepsilon,\mu)=\mathcal{B}_p^{\mathrm{I}}(pm,m,pm,\alpha,\varepsilon',\varepsilon,\mu)$$

and

$$\mathcal{B}_p^{\mathrm{II}}(m,\alpha,\varepsilon',\varepsilon,\mu)=\mathcal{B}_p^{\mathrm{II}}(m,m,m,\alpha,\varepsilon',\varepsilon,\mu)\,.$$

Similarly, for integers N and Δ and positive reals α, γ, ε', ε, $\mu>0$ and $0<p\le 1$, we say that a graph $G=(V,E)$ with $V=[N]$ has property $\widehat{\mathcal{D}}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon',\varepsilon,\mu)$ if G contains no member from $\mathcal{B}_p^{\mathrm{I}}(m,\alpha,\varepsilon',\varepsilon,\mu)\cup\mathcal{B}_p^{\mathrm{II}}(m,\alpha,\varepsilon',\varepsilon,\mu)$ with $m=\gamma p^{\Delta-2}N$ as a (not necessarily induced) subgraph. Next we prove that $G(N,p)$ has property $\widehat{\mathcal{D}}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon',\varepsilon,\mu)$ with high probability.

Proposition 16. *For an integer $\Delta \geq 2$ and positive reals α, ε' and $\mu > 0$ there exists $\varepsilon > 0$ such that for every $\gamma > 0$ there exists $C > 1$ such that if $p > C(\log N/N)^{1/\Delta}$, then $\mathbb{P}(G(N,p) \in \widehat{\mathcal{D}}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon',\varepsilon,\mu)) = 1 - o(1)$.*

Proof. Below we shall only show that a.a.s. $G(N,p)$ contains no subgraphs from $\mathcal{B}^{\mathrm{II}}_p$. The proof for graphs from $\mathcal{B}^{\mathrm{I}}_p$ is analogous.

Let Δ, α, ε', and μ be given. We set

$$\beta = \left(\frac{1}{4}\right)^{4/\mu} \frac{\alpha^2}{4\mathrm{e}^2} \left(\frac{1}{\mathrm{e}}\right)^{4/(\alpha\mu)}$$

and let ε_0 and L be given by Corollary 8 applied with α, β, and ε'. We fix

$$\varepsilon = \min\{\alpha/2\,, \mu/4\,, \varepsilon_0\}\,,$$

and for every $\gamma > 0$ we let $C = 1$. (In fact, any choice of $C > 0$ would suffice for the proof presented here, which concerns only subgraphs from $\mathcal{B}^{\mathrm{II}}_p(m,\alpha,\varepsilon',\varepsilon,\mu)$. For $\mathcal{B}^{\mathrm{I}}_p$ a more careful choice of C is required.)

Suppose $T = (X\dot\cup Y\dot\cup Z, E_T)$ is a tripartite graph from $\mathcal{B}^{\mathrm{II}}_p(m,\alpha,\varepsilon',\varepsilon,\mu)$. We shall find a subgraph of T that, as we shall show, is unlikely to appear in $G(N,p)$. Because of the assumption on T, the bipartite subgraphs $T[X,Y]$, $T[X,Z]$, and $T[Y,Z]$ of T contain at least $(\alpha-\varepsilon)pm^2$ edges each. Furthermore, there is a set $X' \subseteq X$ with $|X'| \geq \mu|X|$ such that for every $x \in X'$ the pair $(N_T(x)\cap Y, N_T(x)\cap Z)$ is not (ε',α,p)-dense. Set

$$X'' = \{x \in X' \colon \; |N_T(x)\cap Y| \geq \alpha pm/2 \text{ and } |N_T(x)\cap Z| \geq \alpha pm/2\}\,.$$

From the (ε,α,p)-denseness of $T[X,Y]$ and $T[X,Z]$ we infer that

$$|X''| \geq (1-2\varepsilon/\mu)|X'| \geq |X'|/2 \geq \mu m/2\,.$$

Fix $x \in X''$. An easy averaging argument shows that there are sets $Y'_x \subseteq N_T(x)\cap Y$ and $Z'_x \subseteq N_T(x)\cap Z$ of size $\varepsilon'\alpha pm/2$ each such that $d_{T,p}(Y'_x,Z'_x) < \alpha-\varepsilon'$. Now let Y_x and Z_x be such that $Y'_x \subseteq Y_x \subseteq N_T(x)\cap Y$ and $Z'_x \subseteq Z_x \subseteq N_T(x)\cap Z$ and $|Y_x| = |Z_x| = \alpha pm/2$. Then, clearly, $T[Y_x,Z_x]$ is not (ε',α,p)-dense. We may thus find a family of pairs $\{(Y_x,Z_x)\colon\; x \in X''\}$ that are not (ε',α,p)-dense. We shall show that such a configuration is unlikely to occur in $G(N,p)$.

Indeed we can fix the sets X'', Y, Z and the edges of the bipartite graph $T[Y,Z]$ in at most

$$\sum_{t\geq(\alpha-\varepsilon)pm^2} \binom{N}{m}^3 \binom{m^2}{t}$$

ways. Since $m = \gamma p^{\Delta-2}N$ (see the definition of $\widehat{\mathcal{D}}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon',\varepsilon,\mu)$) and $p^\Delta N > C\log n > 2L/(\alpha\gamma)$ for sufficiently large N we have $\alpha pm/2 \geq L/p$ and, hence, we can apply Corollary 8 and infer that there are at most

$$\left(\beta^{\alpha pm/2}\binom{m}{\alpha pm/2}^2\right)^{\mu m/2}$$

possibilities for choosing all pairs (Y_x,Z_x) for $x \in X''$. Combining the two estimates above we infer that the probability that such a configuration appears in $G(N,p)$ is bounded from above by

$$\sum_{t\geq(\alpha-\varepsilon)pm^2}\binom{N}{m}^3\binom{m^2}{t}p^t\times\left(\beta^{\alpha pm/2}\binom{m}{\alpha pm/2}^2\right)^{\mu m/2}p^{\mu\alpha pm^2/2}$$
$$\leq\sum_{t\geq(\alpha-\varepsilon)pm^2}\left(\frac{N\mathrm{e}}{m}\right)^{3m}\left(\frac{pm^2\mathrm{e}}{t}\right)^t\times\left(\sqrt{\beta}\frac{2\mathrm{e}}{\alpha}\right)^{\mu\alpha pm^2/2}$$
$$\leq m^2\left(\frac{N\mathrm{e}}{m}\right)^{3m}\left(\mathrm{e}^{1/\alpha}\left(\frac{2\mathrm{e}}{\alpha}\right)^{\mu/2}\beta^{\mu/4}\right)^{\alpha pm^2},$$

where, for the last inequality, we used the fact that the function $f(t)=(pm^2\mathrm{e}/t)^t$ is maximized for $t=pm^2$. Finally, we note that the right-hand side of the last inequality tends to 0 as $N\to\infty$, since $\mathrm{e}^{1/\alpha}(2\mathrm{e}/\alpha)^{\mu/2}\beta^{\mu/4}=1/4$ (owing to the choice of β) and $pm^2\gg m\log N$ (owing to the choice of p and m). □

Now we deduce Proposition 15 from Proposition 16.

Proof of Proposition 15. In order to prove Proposition 15 we need to strengthen Proposition 16 and consider the families $\mathcal{B}_p^{\mathrm{I}}$ and $\mathcal{B}_p^{\mathrm{II}}$ with more general parameters m_1, m_2, and m_3. We shall show that, perhaps surprisingly, this more general statement follows from the "weaker" Proposition 16. Indeed, roughly speaking, we show that each "bad" tripartite graph $T\in\mathcal{B}_p^{\mathrm{II}}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ with arbitrary m_1, m_2, $m_3\geq m$ contains a subgraph $\widehat{T}\in\mathcal{B}_p^{\mathrm{II}}(m,\alpha,\varepsilon'/2,\widehat{\varepsilon},\mu/4)$ for some appropriate $\widehat{\varepsilon}$. The following deterministic statement makes this precise.

Claim 17. *For an integer $\Delta\geq 2$ and positive reals α, ε', μ, and $\widehat{\varepsilon}$ there exists $\varepsilon>0$ such that for every $\gamma>0$ there exists $C>1$ and N_0 such that if $N\geq N_0$ and $p>C(\log N/N)^{1/\Delta}$, then every tripartite graph $T=(X\dot\cup Y\dot\cup Z,E_T)\in\mathcal{B}_p^{\mathrm{II}}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ with $\min\{m_1,m_2,m_3\}\geq m=\gamma p^{\Delta-2}N$ contains a subgraph $\widehat{T}\in\mathcal{B}_p^{\mathrm{II}}(m,\alpha,\varepsilon'/2,\widehat{\varepsilon},\mu/4)$.*

The same claim holds for $\mathcal{B}_p^{\mathrm{I}}$ (and, in fact, the proof is a little simpler), but we only focus on $\mathcal{B}_p^{\mathrm{II}}$ here. Before we prove Claim 17, we note that that claim, combined with Proposition 16, yields Proposition 15, as Proposition 16 guarantees that with probability $1-o(1)$ the random graph $G(N,p)$ contains no such $\widehat{T}$ from $\mathcal{B}_p^{\mathrm{I}}(m,\alpha,\varepsilon'/2,\widehat{\varepsilon},\mu/4)\cup\mathcal{B}_p^{\mathrm{II}}(m,\alpha,\varepsilon'/2,\widehat{\varepsilon},\mu/4)$. □

Proof of Claim 17. Let $\Delta\geq 2$ and α, ε', μ, and $\widehat{\varepsilon}$ be given. We let ε_1, ε_2, and ε be as given by Theorem 7, so that for $\beta=1/2$ we have $\varepsilon_2=\varepsilon_0(\alpha,\beta,\widehat{\varepsilon})$, $\varepsilon_1=\varepsilon_0(\alpha,\beta,\varepsilon_2)$, and $\varepsilon=\min\{\varepsilon_0(\alpha,\beta,\varepsilon_1),\alpha/2,\mu/4\}$. Now for any given γ let $C\geq L/\gamma$, where $L=\max\{L_1,L_2,L_3\}$ and L_1, L_2, and L_3 are given by the three applications of Theorem 7 referred to above. Moreover, let C be sufficiently large, so that the asymptotic estimates in the calculations below become valid. Finally, let $\delta=\varepsilon'/8$.

Let $T=(X\dot\cup Y\dot\cup Z,E_T)\in\mathcal{B}_p^{\mathrm{II}}(m_1,m_2,m_3,\alpha,\varepsilon',\varepsilon,\mu)$ be given. Hence, there exists a set $X'\subseteq X$ with $|X'|\geq\mu|X|$ such that for every $x\in X'$ the pair $(N_T(x)\cap Y,N_T(x)\cap Z)$ is not (ε',α,p)-dense. We consider the set

$$X''=\{x\in X'\colon\ |N_T(x)\cap Y|\geq\alpha pm_2/2\text{ and }|N_T(x)\cap Z|\geq\alpha pm_3/2\}.$$

Owing to the choice of $\varepsilon \le \mu/4$, we infer from the (ε,α,p)-denseness of $T[X,Y]$ and $T[X,Z]$ that $|X''| \ge \mu m_1/2$.

Let each of $\widehat{X} \in \binom{X}{m}$, $\widehat{Y} \in \binom{Y}{m}$, and $\widehat{Z} \in \binom{Z}{m}$ be chosen uniformly at random and let $\widehat{T} = T[\widehat{X},\widehat{Y},\widehat{Z}]$. We shall show that with positive probability $\widehat{T} \in \mathcal{B}_p^{\rm II}(m,\alpha,\varepsilon'/2,\widehat{\varepsilon},\mu/4)$.

By Theorem 7, with probability at least $1-2\beta^m$ the pairs $(\widehat{X},Y)$ and $(\widehat{X},Z)$ are (ε_1,α,p)-dense. Applying Theorem 7 again, we infer that with probability at least $1-4\beta^m$ the pairs $(\widehat{X},\widehat{Y})$, $(\widehat{X},Z)$, and $(\widehat{Y},Z)$ are (ε_2,α,p)-dense and yet another application finally yields that with probability at least $1-6\beta^m$ the pairs

$$(\widehat{X},\widehat{Y}),\ (\widehat{X},\widehat{Z}),\text{ and }(\widehat{Y},\widehat{Z})\text{ are }(\widehat{\varepsilon},\alpha,p)\text{-dense,} \tag{7}$$

which is property (*a*) of part (II) in Definition 13. Below we shall verify that property (*b*) also holds with high probability.

The concentration of the hypergeometric distribution tells us that, with probability at least $1-\exp(-\Omega(m))$, if we set $\widehat{X}'' = \widehat{X} \cap X''$, we have

$$|\widehat{X}''| \ge \frac{1}{4}\mu m. \tag{8}$$

Similarly, with probability at least $1-m\exp(-\Omega(pm))$, we have, for every $x \in \widehat{X}''$, that

$$|N_{\widehat{T}}(x)\cap\widehat{Y}| = (1\pm\delta)\frac{|N_T(x)\cap Y|}{m_2}m = \Omega(pm) \tag{9}$$

and

$$|N_{\widehat{T}}(x)\cap\widehat{Z}| = (1\pm\delta)\frac{|N_T(x)\cap Z|}{m_3}m = \Omega(pm). \tag{10}$$

Recall that for every $x \in \widehat{X}'' \subseteq X'$ there exist sets $Y_x \subseteq N_T(x)\cap Y$ and $Z_x \subseteq N_T(x)\cap Z$ of size at least $\varepsilon'|N_T(x)\cap Y| \ge \varepsilon'\alpha p m_2/2$ and $\varepsilon'|N_T(x)\cap Z| \ge \varepsilon'\alpha p m_3/2$, respectively, such that

$$d_{T,p}(Y_x,Z_x) < \alpha - \varepsilon'. \tag{11}$$

As before, applying the concentration of the hypergeometric distribution, we obtain that, with probability at least $1-m\exp(-\Omega(pm))$, we have, for every $x\in\widehat{X}''$, that

$$|Y_x\cap\widehat{Y}| = (1\pm\delta)\frac{|Y_x|}{m_2}m = \Omega(pm) \tag{12}$$

and

$$|Z_x\cap\widehat{Z}| = (1\pm\delta)\frac{|Z_x|}{m_3}m = \Omega(pm). \tag{13}$$

Below we shall show that, with probability $1-o(1/m)$, for any given $x\in\widehat{X}''$, the pair $(N_{\widehat{T}}(x)\cap\widehat{Y}, N_{\widehat{T}}(x)\cap\widehat{Z})$ is not $(\varepsilon'/2,\alpha,p)$-dense. Summing the failure probability $o(1/m)$ over all choices of x, we deduce that $\widehat{T} = T[\widehat{X},\widehat{Y},\widehat{Z}] \in \mathcal{B}_p^{\rm II}(m,\alpha,\varepsilon'/2,\widehat{\varepsilon},\mu/4)$ with probability $1-o(1)$ (recall (7)).

Fix $x\in\widehat{X}''$. Below, we may and shall assume that (9), (10), (12), and (13) hold. Let $\zeta = (1\pm\delta)|Z_x|m/m_3 = \Omega(pm)$. In what follows, we shall consider the conditional space in which $|Z_x\cap\widehat{Z}| = \zeta$. To remind ourselves of this conditioning, we shall write $\mathbb{P}_\zeta$ and $\mathbb{E}_\zeta$ to denote the probability and the expectation in this space.

For all $y \in Y_x$, let $\Gamma(y) = N_T(y) \cap Z_x$ and set $d_y = |\Gamma(y)|$. We have $\mathbb{E}_\zeta(|\Gamma(y) \cap \widehat{Z}|) = d_y\zeta/|Z_x|$. Suppose now that $d_y \geq (\varepsilon'/20\mathrm{e})p|Z_x|$. Then

$$\mathbb{P}_\zeta\left(|\Gamma(y) \cap \widehat{Z}| \geq (1+\delta)\frac{d_y}{|Z_x|}\zeta\right) \leq \exp\left(-\frac{1}{3}\delta^2\frac{d_y}{|Z_x|}\zeta\right) \leq \exp\left(-\frac{1}{3}\delta^2\frac{\varepsilon'}{20\mathrm{e}}p\zeta\right) = \exp\left(-\Omega(p^2m)\right). \quad (14)$$

Consider now the case in which $d_y < (\varepsilon'/20\mathrm{e})p|Z_x|$. Then

$$\begin{aligned}\mathbb{P}_\zeta\left(|\Gamma(y) \cap \widehat{Z}| \geq \frac{d_y}{|Z_x|}\zeta + \frac{\varepsilon'}{10}p\zeta\right) &\leq \mathbb{P}_\zeta\left(|\Gamma(y) \cap \widehat{Z}| \geq \frac{\varepsilon'}{10}p\zeta\right)\\ &\leq \left(\frac{\mathrm{e}}{(\varepsilon'/10)p\zeta}\frac{d_y}{|Z_x|}\zeta\right)^{(\varepsilon'/10)p\zeta}\\ &\leq \left(\frac{\mathrm{e}}{(\varepsilon'/10)p}(\varepsilon'/20\mathrm{e})p\right)^{(\varepsilon'/10)p\zeta}\\ &= \left(\frac{1}{2}\right)^{(\varepsilon'/10)p\zeta}\\ &= \exp\left(-\Omega(p^2m)\right). \quad (15)\end{aligned}$$

Let us note that, if $d_y < (\varepsilon'/20\mathrm{e})p|Z_x|$, then

$$\frac{d_y}{|Z_x|}\zeta + \frac{\varepsilon'}{10}p\zeta \leq \frac{\varepsilon'}{20\mathrm{e}}p\zeta + \frac{\varepsilon'}{10}p\zeta \leq \frac{1}{8}\varepsilon' p\zeta. \quad (16)$$

Because of (11), (14), (15), and (16), we have, with probability $1 - o(1/m)$, that

$$\begin{aligned}e(Y_x, Z_x \cap \widehat{Z}) &\leq \sum_{y\in Y_x}(1+\delta)\frac{d_y}{|Z_x|}\zeta + \sum_{y \in Y_x}\frac{1}{8}\varepsilon' p\zeta\\ &= (1+\delta)\frac{\zeta}{|Z_x|}\sum_{y\in Y_x} d_y + \frac{1}{8}\varepsilon' p\zeta|Y_x|\\ &\leq (1+\delta)\frac{\zeta}{|Z_x|}(\alpha - \varepsilon')p|Y_x||Z_x| + \frac{1}{8}\varepsilon' p\zeta|Y_x|,\end{aligned}$$

whence, recalling that $|Z_x \cap \widehat{Z}| = \zeta$,

$$d_{T,p}(Y_x, Z_x \cap \widehat{Z}) \leq (1+\delta)(\alpha - \varepsilon') + \frac{1}{8}\varepsilon' \leq \alpha - \frac{1}{4}\varepsilon'. \quad (17)$$

Repeating the same argument with Y_x replaced with $Z_x \cap \widehat{Z}$ and with Z_x replaced with Y_x, we obtain that, with probability $1 - o(1/m)$,

$$d_{T,p}(Y_x \cap \widehat{Y}, Z_x \cap \widehat{Z}) \leq \alpha - \frac{1}{2}\varepsilon'.$$

This concludes the proof of Claim 17. □

4. Ramsey universal graphs

4.1. **Proof of the main result.** In this section we prove Theorem 3, namely, we show that for $p = p(N) \geq C(\log N/N)^{1/\Delta}$ the random graph $G(N,p)$ is partition universal for $\mathcal{H}_{\Delta,n}$ for n of the form $\lfloor cN \rfloor$ for some $c > 0$. In view of the results from Section 3 this follows directly from the following deterministic statement.

Lemma 18. *For every $\Delta \geq 2$ there exist $\widetilde{\Delta} \geq 2$ and positive constants μ, α, $\varepsilon_0, \dots, \varepsilon_{\widetilde{\Delta}}$, ξ, γ, B, and n_0 such that for every $n \geq n_0$ the following holds. If $G = (V, E)$ is a graph on $V = [N]$, where $N \geq Bn$, such that for some $0 < p \leq 1$ we have*

(*i*) $G \in \mathcal{U}_{N,p}$,
(*ii*) $G \in \mathcal{E}^k_{N,p}(\xi)$ *for every* $k = 1, \dots, \Delta$, *and*
(*iii*) $G \in \mathcal{D}^{\Delta}_{N,p}(\gamma, \alpha, \varepsilon_k, \varepsilon_{k-1}, \mu)$ *for every* $k = 1, \dots, \widetilde{\Delta}$,

then G is partition universal for $\mathcal{H}_{\Delta,n}$.

Before we prove Lemma 18, we deduce Corollary 19 below, which implies Theorem 3 immediately.

Corollary 19. *For every $\Delta \geq 2$ there exist B and $C > 0$ such that, if $p = p(N) \geq C(\log N/N)^{1/\Delta}$ and $n = n(N) = \lfloor N/B \rfloor$, then*

$$\mathbb{P}\big(G(N,p) \text{ is partition universal for } \mathcal{H}_{\Delta,n}\big) = 1 - o(1)\,. \tag{18}$$

Proof. For a given $\Delta \geq 2$, let $\widetilde{\Delta}$, μ, α, $\varepsilon_0, \dots, \varepsilon_{\widetilde{\Delta}}$, ξ, γ, and B be given by Lemma 18. Then let C be large enough so that Proposition 12 holds for every $k = 1, \dots, \Delta$, and ξ and so that Proposition 15 holds for every $k = \widetilde{\Delta}, \dots, 1$ with μ, α, $\varepsilon'_k = \varepsilon_k$, and $\varepsilon_{k-1} = \varepsilon(\Delta - 1, \alpha, \varepsilon' = \varepsilon_k, \mu)$ from Proposition 15. Consequently, with probability $1 - o(1)$, the random graph $G(N,p)$ satisfies properties (*ii*)–(*iii*) of Lemma 18 due to Propositions 12 and 15. Finally, property (*i*) holds with probability $1 - o(1)$ by Proposition 10 as $\Delta \geq 2$. Thus (18) follows. □

4.2. **Proof of the main technical lemma.** In this section we prove the main technical lemma, Lemma 18. The proof follows the strategy in the proof of Chvátal et al. in [10], but includes ideas from [5] and [32], and is based on the sparse regularity lemma.

Proof of Lemma 18. The proof consists of four parts. In the first part we fix all constants needed in the proof. In the second part we consider the given graph G along with a fixed 2-coloring of its edges. We have to show that G contains a monochromatic $\mathcal{H}_{\Delta,n}$-universal graph. In other words, we have to embed every graph $H \in \mathcal{H}_{\Delta,n}$ into one of the two monochromatic subgraphs of G. To that end, we first prepare the graph G and here the sparse regularity lemma will be the key tool. In the third part we shall prepare a given graph $H \in \mathcal{H}_{\Delta,n}$ for the embedding. In the last part we then embed H into a monochromatic subgraph of G.

Constants. Let $\Delta \geq 2$ be an integer. We first fix

$$\widetilde{\Delta} = \Delta^4 + 2\Delta + 1$$

and we set

$$r = R(\widetilde{\Delta}, \widetilde{\Delta})\,,$$

where $R(\widetilde{\Delta}, \widetilde{\Delta})$ is the Ramsey number that guarantees that every 2-coloring of the edges of the complete graph K_r yields a monochromatic copy of $K_{\widetilde{\Delta}}$. Next we define the constants μ, α, $\varepsilon_0, \dots, \varepsilon_{\widetilde{\Delta}}$, ξ, γ, B, and n_0 of Lemma 18. First we set

$$\mu = \frac{1}{4\Delta^2} \quad \text{and} \quad \alpha = \frac{1}{4}\,, \tag{19}$$

and we fix ε_k for $k = \widetilde{\Delta}, \widetilde{\Delta} - 1, \dots, 0$ by setting

$$\varepsilon_{\widetilde{\Delta}} = \frac{1}{12\widetilde{\Delta}} \quad \text{and} \quad \varepsilon_{k-1} = \min\left\{\varepsilon(\Delta - 1, \alpha, \varepsilon' = \varepsilon_k, \mu),\, \varepsilon_k\right\} \text{ for } k = \widetilde{\Delta}, \dots, 1\,, \tag{20}$$

where $\varepsilon(\Delta - 1, \alpha, \varepsilon' = \varepsilon_k, \mu)$ is given by Proposition 15.

Next we set

$$\varepsilon = \min\left\{\frac{\varepsilon_0}{2}, \frac{1}{2(r-1)}\right\}, \quad K = 2\,, \quad \text{and} \quad t_0 = 2r \tag{21}$$

and let T_0, η, and N_0 be the constants guaranteed by the sparse regularity lemma, Theorem 5, for ε, K, and t_0 given above. Finally, we set

$$\gamma = \frac{1-\varepsilon}{4^{\Delta-1}T_0}\,, \qquad \xi = \frac{1}{7 \cdot 4^{\Delta+1} \cdot T_0}\,, \qquad B = \frac{1}{\xi}\,, \tag{22}$$

and

$$n_0 = \max\left\{\frac{N_0}{B}, \frac{1}{\eta^2}, T_0^2, 2^{4/\varepsilon_0}, e^{1/\eta}\right\}. \tag{23}$$

This concludes the definition of the constants involved in the proof of Lemma 18.

Preparing G. Now let $n \geq n_0$ be given and let $G = (V, E)$ be a graph on $V = [N]$, where $N \geq Bn \geq N_0$, satisfies assumptions (i)–(iii) of Lemma 18 for some $0 < p \leq 1$. We fix an arbitrary coloring of the edges $E = E_R \dot\cup E_B$ of G with two colors, say red and blue, and let $G_R = (V, E_R)$ and $G_B = (V, E_B)$ be the corresponding monochromatic subgraphs. We have to show that one of G_R or G_B will contain every H in $\mathcal{H}_{\Delta,n}$. To that end, first use the sparse regularity lemma to "locate" an appropriate "regular" subgraph in either G_R or G_B.

More precisely, we apply the regularity lemma with $\varepsilon = \min\{\varepsilon_0/2, 1/(r-1)\}$, $K = 2$, $t_0 = 2r$, and p to G_R. Note that, owing to property (i) of Lemma 18 (see Definition 9), the graph G is $(1/\log N, 1 + 1/\log N)$-bounded. Since $G_R \subseteq G$, $1/\log N \leq 1$, and $N/\log N \leq \eta N$ (because of the choice of n_0 in (23)) we infer that indeed G_R is (η, K)-bounded (see (21)). Consequently, Theorem 5 yields an (ε, t)-equitable (ε, G_R, p)-regular partition $V_0 \dot\cup V_1 \dot\cup \dots V_t$ of V with $t_0 \leq t \leq T_0$.

We consider an auxiliary graph A with vertex set $[t] = \{1, \dots, t\}$ and $\{i, j\}$ being an edge if and only if the pair (V_i, V_j) is (ε, p)-regular for G_R. Since the partition $V_0 \dot\cup V_1 \dot\cup \dots V_t$ is (ε, G_R, p)-regular, at most $\varepsilon\binom{t}{2} \leq \frac{1}{2(r-1)}\binom{t}{2} < (r-1)\binom{t/(r-1)}{2}$ of the pairs of the auxiliary graph are missing and hence, by Turán's theorem, A contains a clique K_r with r vertices. In other words, there exists an index set $I_r = \{i_1, \dots, i_r\} \subseteq [t]$ such that (V_i, V_j) is (ε, p)-regular for G_R for all $\{i, j\} \in \binom{I_r}{2}$. Moreover, since $G \in \mathcal{U}_{N,p}$ and since $1/\log N \leq N/T_0$ (see (23)) it follows directly from the definition of (ε, p)-regularity that (V_i, V_j) is $(\varepsilon + 2/\log N, p)$-regular for the graph G_B. Because of (21) and (23), we have $\varepsilon + 2/\log N \leq \varepsilon_0/2 + \varepsilon_0/2$ and, hence, (V_i, V_j) is (ε_0, p)-regular for G_R and for G_B for all $\{i, j\} \in \binom{I_r}{2}$.

Next we color the edges of the clique $K_r \subseteq A$ red and blue. We color an edge $\{i, j\} \in \binom{I_r}{2}$ red if $d_{G_R,p}(V_i, V_j) \geq d_{G_B,p}(V_i, V_j)$ and blue otherwise. Note that, again from the fact that $G \in \mathcal{U}_{N,p}$ and $1/\log N \leq N/T_0$ we infer that $d_{G_R,p}(V_i, V_j) + d_{G_B,p}(V_i, V_j) \geq 1 - 1/\log N$ and, therefore,

$$\max\left\{d_{G_R,p}(V_i, V_j), d_{G_B,p}(V_i, V_j)\right\} \geq \frac{1}{2} - \frac{1}{2\log N} \geq \frac{1}{3}$$

for every $\{i, j\} \in \binom{[t]}{2}$.

Because of the choice of $r \geq R(\widetilde{\Delta}, \widetilde{\Delta})$ there exists a monochromatic clique $K_{\widetilde{\Delta}} \subseteq K_r \subseteq A$ on $\widetilde{\Delta}$ vertices. Let $J \subseteq I$ be the vertex set of the monochromatic clique $K_{\widetilde{\Delta}}$. Summarizing, the above ensures the existence of a set $J \subseteq I$ of cardinality $\widetilde{\Delta}$ such that either

$$(V_i, V_j) \text{ is } (\varepsilon_0, p)\text{-regular for } G_R \text{ and } d_{G_R,p}(V_i, V_j) \geq 1/3 \text{ for all } \{i,j\} \in \binom{J}{2} \quad (24)$$

or the same statement holds for G_B. Without loss of generality we assume that (24) holds and we shall show that G_R induced on $\bigcup_{i \in J} V_i$ will contain any H from $\mathcal{H}_{\Delta,n}$.

Preparing H. Fix some $H = (W, F) \in \mathcal{H}_{\Delta,n}$. We consider the third power $H^3 = (W, F^3)$ of H, i.e., $\{w, w'\} \in F^3$ if and only if $w \neq w'$ and there exists a w–w'-path with at most three edges in H. Since $\Delta(H) \leq \Delta$ we have

$$\Delta(H^3) \leq \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 = \Delta^3 - \Delta^2 + \Delta$$

and consequently $\chi(H^3) \leq \Delta^3 - \Delta^2 + \Delta + 1$. Fix a $(\Delta^3 - \Delta^2 + \Delta + 1)$-vertex coloring f of H^3 with colors $1, \dots, \Delta^3 - \Delta^2 + \Delta + 1$. This way we obtain a partition of W into $\Delta^3 - \Delta^2 + \Delta + 1$ classes such that if two vertices w and w' are elements of the same class, then their distance in H is at least four; in particular, there are no edges between $N_H(w)$ and $N_H(w')$, since otherwise $\{w, w'\}$ would be an edge in H^3. We now refine the partition induced by the color classes of f according to the "left-degrees" of the vertices. More precisely, we say two vertices w and w' are equivalent if $f(w) = f(w')$ and

$$\left| N_H(w) \cap \{x \in W \colon f(x) < f(w)\} \right| = \left| N_H(w') \cap \{x \in W \colon f(x) < f(w')\} \right|,$$

i.e., w and w' are equivalent if they have the same color in f and the same number of neighbours with colors of smaller number. Clearly, this equivalence relation partitions W into at most $(\Delta^3 - \Delta^2 + \Delta + 1)(\Delta + 1) = \widetilde{\Delta}$ classes. Denote the partition classes by $W_1, \dots, W_{\widetilde{\Delta}}$ (allowing empty classes if necessary) and let $g \colon W \to [\widetilde{\Delta}]$ be the corresponding partition function, i.e.,

$$g(w) = j \quad \text{if and only if} \quad w \in W_j.$$

Thus, if $g(w) = g(w')$, then $\left| N_H(w) \cap \{x \in W \colon g(x) < g(w)\} \right| = \left| N_H(w') \cap \{x \in W \colon g(x) < g(w')\} \right|$. For an integer $\ell \leq g(w)$ we denote by

$$\mathrm{ldeg}_g^{\ell}(w) := \left| N_H(w) \cap \{x \in W \colon g(x) \leq \ell\} \right|$$

the *left-degree of w with respect to g and ℓ.*

Embedding of H into G. After the preparation of G and H we are able to embed H into G_R. We may relabel the vertex classes V_i of G_R with $i \in J$ and assume $J = [\widetilde{\Delta}]$. We proceed inductively and embed the vertex class W_ℓ into V_ℓ one at a time, for $\ell = 1, \dots, \widetilde{\Delta}$. To this end, we verify the following statement $(\mathcal{S}_\ell)$ for $\ell = 0, \dots, \widetilde{\Delta}$.

$(\mathcal{S}_\ell)$ There exists a partial embedding φ_ℓ of $H[\bigcup_{j=1}^{\ell} W_j]$ into $G_R[\bigcup_{j=1}^{\ell} V_j]$ such that for every $z \in \bigcup_{j=\ell+1}^{\widetilde{\Delta}} W_j$ there exists a *candidate set* $C_\ell(z) \subseteq V(G)$ given by
 (*a*) $C_\ell(z) = \bigcap \{N_{G_R}(\varphi_\ell(x)) \colon x \in N_H(z) \text{ and } g(x) \leq \ell\} \cap V_{g(z)}$,
 satisfying
 (*b*) $|C_\ell(z)| \geq (p/4)^{\mathrm{ldeg}_g^{\ell}(z)} m$, where $m = |V_{g(z)}| \geq (1 - \varepsilon)N/t$, and

(c) for every edge $\{z, z'\} \in F = E(H)$ with $g(z), g(z') > \ell$ the pair $\big(C_\ell(z), C_\ell(z')\big)$ is $(\varepsilon_\ell, 1/3 - \ell\varepsilon_{\widetilde{\Delta}}, p)$-dense in G_R.

Remark 20. *In what follows we shall use the following convention. Vertices from G_R will be denoted by v and vertices from H will be usually named w. However, since the embedding of H into G will be divided into $\widetilde{\Delta}$ rounds, we shall find it convenient to distinguish among the vertices of H. We shall use the letter x for vertices that have already been embedded, the letter y for vertices that will be embedded in the current round, while z will denote vertices that we shall embed at a later step.*

Statement $(\mathcal{S}_\ell)$ ensures the existence of a partial embedding of the first ℓ classes $W_1, \dots, W_\ell$ of H such that for every unembedded vertex z there exists a candidate set $C_\ell(z)$ that is not too small (see part (b)). Moreover, if we embed z into its candidate set, then its image will be adjacent to all vertices $\varphi_\ell(x)$ with $x \in (W_1 \cup \dots \cup W_\ell) \cap N_H(z)$ (see part (a)). The last property, part (c), says that edges of H for which none of the endvertices are embedded already the respective candidate sets induce (ε, α, p)-dense pairs. This property will be crucial for the inductive proof.

Before we verify $(\mathcal{S}_\ell)$ for $\ell = 0, \dots, \widetilde{\Delta}$ by induction on ℓ we note that $(\mathcal{S}_{\widetilde{\Delta}})$ implies that H can be embedded into G_R. Since H was an arbitrary graph from $\mathcal{H}_{\Delta,n}$ and we fixed an arbitrary coloring of the edges of G, this implies $G \longrightarrow H$ for every $H \in \mathcal{H}_{\Delta,n}$. Consequently, verifying $(\mathcal{S}_\ell)$ yields the proof of Lemma 18.

Basis of the induction: $\ell = 0$. We first verify $(\mathcal{S}_0)$. In this case φ_0 is the empty mapping and for every $z \in W$ we have, according to (a), $C_0(z) = V_{g(z)}$, as there is no vertex $x \in N_H(z)$ with $g(x) \le 0$. Also, property (b) holds by definition of $C_0(z)$ for every $z \in W$. Finally, property (c) follows from the property that (V_i, V_j) is (ε_0, p)-regular for G_R and, consequently, $\big(C_0(z), C_0(z')\big)$ is $(\varepsilon_0, 1/3, p)$-dense in G_R for every edge $\{z, z'\}$ of H (see (24)).

Induction step: $\ell \to \ell + 1$. For the inductive step, we suppose that $\ell < \widetilde{\Delta}$ and assume that statement $(\mathcal{S}_\ell)$ holds; we have to construct $\varphi_{\ell+1}$ with the required properties. Our strategy is as follows. In the first step, we find for every $y \in W_{\ell+1}$ an appropriate subset $C(y) \subseteq C_\ell(y)$ of the candidate set such that if $\varphi_{\ell+1}(y)$ is chosen from $C(y)$, then (*) the new candidate set $C_{\ell+1}(z) := C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))$ of every "right-neighbour" z of y will not shrink too much and (**) property (c) will continue to hold.

Note, however, that in general $|C(y)| \le |C_\ell(y)| = o(N) \ll |W_{\ell+1}|$ (if $\mathrm{ldeg}_g^\ell \ge 1$) and, hence, we cannot "blindly" select $\varphi_{\ell+1}(y)$ from $C(y)$. Instead, in the second step, we shall verify Hall's condition to find a system of distinct representatives for the family $\{C(y)\colon\ y \in W_{\ell+1}\}$ and we let $\varphi_{\ell+1}(y)$ be the representative of $C(y)$. (A similar idea was used in [5, 32].) We now give the details of those two steps.

For the first step, fix $y \in W_{\ell+1}$ and set

$$N_H^{\ell+1}(y) := \{z \in N_H(y)\colon\ g(z) > \ell + 1\}\,.$$

A vertex $v \in C_\ell(y)$ will be "bad" (i.e., we shall not select v for $C(y)$) if there exists a vertex $z \in N_H^{\ell+1}(y)$ for which $N_{G_R}(v) \cap C_\ell(z)$, in view of (b) and (c) of $(\mathcal{S}_{\ell+1})$, cannot play the rôle of $C_{\ell+1}(z)$.

We first prepare for (b) of $(\mathcal{S}_{\ell+1})$. Fix a vertex $z \in N_H^{\ell+1}(y)$. Since $(C_\ell(y), C_\ell(z))$ is an $(\varepsilon_\ell, 1/3 - \ell\varepsilon_{\widetilde{\Delta}}, p)$-dense pair, there exist at most $\varepsilon_\ell|C_\ell(y)| \le \varepsilon_{\widetilde{\Delta}}|C_\ell(y)|$ vertices

v in $C_\ell(y)$ such that

$$|N_{G_R}(v)\cap C_\ell(z)| < \big(d_{G_R,p}(C_\ell(y),C_\ell(z)) - \varepsilon_{\widetilde{\Delta}}\big)p|C_\ell(y)|\,.$$

Repeating the above for all $z\in N_H^{\ell+1}(y)$, we infer from (a) and (b) of $(\mathcal{S}_\ell)$, that there are at most $\Delta\varepsilon_{\widetilde{\Delta}}|C_\ell(y)|$ vertices $v\in C_\ell(y)$ such that the following fails to be true for some $z\in N_H^{\ell+1}(y)$:

$$\begin{aligned}|N_{G_R}(v)\cap C_\ell(z)| &\geq \big(d_{G_R,p}(C_\ell(y),C_\ell(z)) - \varepsilon_{\widetilde{\Delta}}\big)p|C_\ell(z)|\\ &\overset{(a),\,(b)}{\geq} \left(\frac{1}{3}-(\ell+1)\varepsilon_{\widetilde{\Delta}}\right)p\left(\frac{p}{4}\right)^{\mathrm{ldeg}_g^{\ell}(z)}|V_{g(z)}| \overset{(20)}{\geq} \left(\frac{p}{4}\right)^{\mathrm{ldeg}_g^{\ell+1}(z)}|V_{g(z)}|\,. \qquad (25)\end{aligned}$$

For property (c) of $(\mathcal{S}_{\ell+1})$, we fix an edge $e=\{z,z'\}$ with $g(z),\,g(z')>\ell+1$ and with at least one end vertex in $N_H^{\ell+1}(y)$. There are at most $\Delta(\Delta-1)<\Delta^2$ such edges. Note that if both vertices z and z' are neighbours of y, i.e., $z,\,z'\in N_H^{\ell+1}(y)$, then

$$\max\big\{\mathrm{ldeg}_g^{\ell}(y),\mathrm{ldeg}_g^{\ell}(z),\mathrm{ldeg}_g^{\ell}(z')\big\}\leq\Delta-2\,,$$

since all three vertices y, z, and z' have at least two neighbours in $W_{\ell+1}\cup\dots\cup W_{\widetilde{\Delta}}$. From property (b) of $(\mathcal{S}_\ell)$ we infer

$$\begin{aligned}&\min\Big\{|C_\ell(y)|\,,|C_\ell(z)|\,,|C_\ell(z')|\Big\}\\ &\qquad\geq\left(\frac{p}{4}\right)^{\max\{\mathrm{ldeg}_g^{\ell}(y),\mathrm{ldeg}_g^{\ell}(z),\mathrm{ldeg}_g^{\ell}(z')\}}(1-\varepsilon)\frac{N}{T_0}\overset{(22)}{\geq}\gamma p^{\Delta-2}N.\end{aligned}$$

Furthermore, $1/3-\ell\varepsilon_{\widetilde{\Delta}}\geq\alpha=1/4$ (see (20)). Hence $G_R\subseteq G$ and $G\in\mathcal{D}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon_{\ell+1},\varepsilon_\ell,\mu)$ imply that there are at most $\mu|C_\ell(y)|$ vertices $v\in C_\ell(y)$ such that the pair $(N_{G_R}(v)\cap C_\ell(z),N_{G_R}(v)\cap C_\ell(z'))$ fails to be $(\varepsilon_{\ell+1},1/3-(\ell+1)\varepsilon_{\widetilde{\Delta}},p)$-dense.

If, on the other hand, say, only $z\in N_H^{\ell+1}(y)$ and $z'\notin N_H^{\ell+1}(y)$, then

$$\max\{\mathrm{ldeg}_g^{\ell}(y),\mathrm{ldeg}_g^{\ell}(z')\}\leq\Delta-1\quad\text{and}\quad\mathrm{ldeg}_g^{\ell}(z)\leq\Delta-2.$$

Consequently, (similarly as above)

$$\min\Big\{|C_\ell(y)|\,,|C_\ell(z')|\Big\}\geq\gamma p^{\Delta-1}N\quad\text{and}\quad|C_\ell(z)|\geq\gamma p^{\Delta-2}N$$

and we can appeal to the fact that $G\in\mathcal{D}^{\Delta}_{N,p}(\gamma,\alpha,\varepsilon_{\ell+1},\varepsilon_\ell,\mu)$ to infer that there are at most $\mu|C_\ell(y)|$ vertices $v\in C_\ell(y)$ such that $(N_{G_R}(v)\cap C_\ell(z),C_\ell(z'))$ fails to be $(\varepsilon_{\ell+1},1/3-(\ell+1)\varepsilon_{\widetilde{\Delta}},p)$-dense. For a given $v\in C_\ell(y)$, let $\widehat{C}_\ell(z)=C_\ell(z)\cap N_{G_R}(v)$ if $z\in N_H^{\ell+1}(y)$ and $\widehat{C}_\ell(z)=C_\ell(z)$ if $z\notin N_H^{\ell+1}(w)$, and define $\widehat{C}_\ell(z')$ analogously.

Summarizing the above we infer that there are at least

$$(1-\Delta\varepsilon_{\widetilde{\Delta}}-\Delta^2\mu)|C_\ell(y)| \qquad (26)$$

vertices $v\in C_\ell(y)$ such that

- (b') $|N_{G_R}(v)\cap C_\ell(z)|\geq(p/4)^{\mathrm{ldeg}_g^{\ell+1}(z)}|V_{g(z)}|$ for every $z\in N_H^{\ell+1}(y)$ (see (25) and
- (c') $(\widehat{C}_\ell(z),\widehat{C}_\ell(z'))$ is $(\varepsilon_{\ell+1},1/3-(\ell+1)\varepsilon_{\widetilde{\Delta}},p)$-dense for all edges $\{z,z'\}$ of H with $g(z),\,g(z')>\ell+1$ and $\{z,z'\}\cap N_H^{\ell+1}(y)\neq\emptyset$.

Let $C(y)$ be the set of those vertices v from $C_\ell(y)$ satisfying properties (b') and (c') above. Recall that $\mathrm{ldeg}_g^{\ell}(y)=\mathrm{ldeg}_g^{\ell}(y')$ for all $y,\,y'\in W_{\ell+1}$ and set

$$k=\mathrm{ldeg}_g^{\ell}(y)\text{ for some }y\in W_{\ell+1}.$$

Since $y \in W_{\ell+1}$ was arbitrary, we infer from (26), the choices of μ and $\varepsilon_{\widetilde{\Delta}}$ in (19) and (20) and property (*b*) of $(\mathcal{S}_\ell)$ that

$$|C(y)| \geq (1 - \Delta\varepsilon_{\widetilde{\Delta}} - \Delta^2\mu)|C_\ell(y)| \geq (1 - \Delta\varepsilon_{\widetilde{\Delta}} - \Delta^2\mu)\left(\frac{p}{4}\right)^k (1-\varepsilon)\frac{N}{T_0} \geq \frac{1}{4^{k+1}}p^k\frac{N}{T_0}\,. \tag{27}$$

We now turn to the aforementioned second part of the inductive step. Here we ensure the existence of a system of distinct representatives for the set system

$$\mathcal{C}_{\ell+1} = \{C(y)\colon\ y \in W_{\ell+1}\}\,.$$

We shall appeal to Hall's condition and show that for every subfamily $\mathcal{C}' \subseteq \mathcal{C}_{\ell+1}$ we have

$$|\mathcal{C}'| \leq \left|\bigcup_{C\in\mathcal{C}'} C\right|. \tag{28}$$

Because of (27), assertion (28) holds for all families $\mathcal{C}'$ with $1 \leq |\mathcal{C}'| \leq 4^{-k-1}p^kN/T_0$.

Thus, consider a family $\mathcal{C}' \subseteq \mathcal{C}_{\ell+1}$ with $|\mathcal{C}'| > 4^{-k-1}p^kN/T_0$. For every $y \in W_{\ell+1}$ we have $\mathrm{ldeg}_g^\ell(y) = k$. Hence, we have a k-tuple $K(y) = \{u_1(y), \dots, u_k(y)\}$ of already embedded vertices of H such that $K(y) = N_H(y) \setminus N_H^{\ell+1}(y)$. Note that for two distinct vertices y, $y' \in W_{\ell+1}$ the sets $K(y)$ and $K(y')$ are disjoint. This follows from the fact that the distance in H between y and y' is at least four and if $K(y) \cap K(y') \neq \emptyset$, then this distance would be at most two. Consequently, the sets $\varphi(K(y))$ and $\varphi(K(y'))$ are disjoint as well and, therefore, $\mathcal{F}_k = \{\varphi(K(y))\colon\ y \in W_{\ell+1}\} \subseteq \binom{V}{k}$ is a family of pairwise disjoint k-sets in V. Moreover,

$$C(y) \subseteq \bigcap_{v\in\varphi(K(y))} N_{G_R}(v) \subseteq \bigcap_{v\in\varphi(K(y))} N_G(v)\,.$$

Let

$$U = \bigcup_{C(y)\in\mathcal{C}'} C(y) \subseteq V_{\ell+1}\,,$$

and suppose for a contradiction that

$$|U| < |\mathcal{C}'| = |\mathcal{F}_k|. \tag{29}$$

We now use property (*ii*) of Lemma 18, namely, $G \in \mathcal{E}^k_{N,p}(\xi)$ applied for $\mathcal{F}_k$ and U. We deduce that

$$e_{\Gamma(k,G)}(\mathcal{F}_k, U) \leq p^k|\mathcal{F}_k||U| + 6\xi Np^k|\mathcal{F}_k|\,.$$

On the other hand, because of (27), we have

$$e_{\Gamma(k,G)}(\mathcal{F}_k, U) \geq \frac{1}{4^{k+1}}p^k\frac{N}{T_0}|\mathcal{F}_k|.$$

Combining the last two inequalities we infer

$$\left|\bigcup_{C(y)\in\mathcal{C}'} C(y)\right| = |U| \geq \left(\frac{1}{4^{k+1}}\frac{1}{T_0} - 6\xi\right)N \overset{(22)}{\geq} \xi N \geq \xi Bn \overset{(22)}{=} n \geq |W_{\ell+1}| \geq |\mathcal{C}'|,$$

which contradicts (29). This contradiction shows that (29) does not hold, that is, Hall's condition (28) does hold. Hence, there exists a system of representatives for $\mathcal{C}_{\ell+1}$, i.e., an injective mapping $\psi\colon W_{\ell+1} \to \bigcup_{y\in W_{\ell+1}} C(y)$ such that $\psi(y) \in C(y)$ for every $y \in W_{\ell+1}$.

Finally, we extend φ_ℓ and define $C_{\ell+1}(z)$ for $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$. For that we set

$$\varphi_{\ell+1}(w) = \begin{cases} \varphi_\ell(w)\,, & \text{if } w \in \bigcup_{j=1}^{\ell} W_j,\\ \psi(w)\,, & \text{if } w \in W_{\ell+1}\,. \end{cases}$$

Note that every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ has at most one neighbour in $W_{\ell+1}$, as otherwise there would be two vertices y and $y' \in W_{\ell+1}$ with distance at most 2 in H, which contradicts the fact that g and f are valid vertex colorings of H^3. Consequently, for every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ we can set

$$C_{\ell+1}(z) = \begin{cases} C_\ell(z)\,, & \text{if } N_H(z) \cap W_{\ell+1} = \emptyset,\\ C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))\,, & \text{if } N_H(z) \cap W_{\ell+1} = \{y\}. \end{cases}$$

In what follows we show that $\varphi_{\ell+1}$ and $C_{\ell+1}(z)$ for every $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ have the desired properties and validate $(\mathcal{S}_{\ell+1})$.

First of all, from (a) of $(\mathcal{S}_\ell)$, combined with $\varphi_{\ell+1}(y) \in C(y) \subseteq C_\ell(y)$ for every $y \in W_{\ell+1}$ and the property that $\{\varphi_{\ell+1}(y) \colon y \in W_{\ell+1}\}$ is a system of distinct representatives, we infer that $\varphi_{\ell+1}$ is indeed a partial embedding of $H[\bigcup_{j=1}^{\ell+1} W_j]$.

Next we shall verify properties (a) and (b) of $(\mathcal{S}_{\ell+1})$. So let $z \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$ be fixed. If $N_H(z) \cap W_{\ell+1} = \emptyset$, then $C_{\ell+1}(z) = C_\ell(z)$, $\operatorname{ldeg}_g^{\ell+1}(z) = \operatorname{ldeg}_g^{\ell}(z)$, which yields (a) and (b) of $(\mathcal{S}_{\ell+1})$ for that case. If, on the other hand, $N_H(z) \cap W_{\ell+1} \neq \emptyset$, then there exists a unique neighbour $y \in W_{\ell+1}$ of H (owing to the fact that g is a refinement of a valid vertex coloring of H^3). Because of the definition of $C_{\ell+1}(z) = C_\ell(z) \cap N_{G_R}(\varphi_{\ell+1}(y))$ part (a) of $(\mathcal{S}_{\ell+1})$ follows in this case. Moreover, since $\varphi_{\ell+1}(y) \in C(y)$, we infer directly from (b') that (b) of $(\mathcal{S}_{\ell+1})$ is satisfied in this case.

Finally, we verify property (c) of $(\mathcal{S}_{\ell+1})$. Let $\{z, z'\}$ be an edge of H with $z, z' \in \bigcup_{j=\ell+2}^{\widetilde{\Delta}} W_j$. We consider three cases, depending on the size of $N_H(z) \cap W_{\ell+1}$ and of $N_H(z') \cap W_{\ell+1}$. If $N_H(z) \cap W_{\ell+1} = \emptyset$ and $N_H(z') \cap W_{\ell+1} = \emptyset$, then part (c) of $(\mathcal{S}_{\ell+1})$ follows directly from part (c) of $(\mathcal{S}_\ell)$ and $\varepsilon_{\ell+1} \geq \varepsilon_\ell$, combined with $C_{\ell+1}(z) = C_\ell(z)$, $C_{\ell+1}(z') = C_\ell(z')$. If $N_H(z) \cap W_{\ell+1} = \{y\}$ and $N_H(z') \cap W_{\ell+1} = \emptyset$, then (c) of $(\mathcal{S}_{\ell+1})$ follows from (c') and the definition of $C_{\ell+1}(z)$ and $C_{\ell+1}(z')$. If $N_H(z) \cap W_{\ell+1} = \{y\}$ and $N_H(z') \cap W_{\ell+1} = \{y'\}$, then $y = y'$, as otherwise there would be a y-y'-path in H with three edges, i.e., $\{y, y'\}$ would be an edge in H^3, which would imply that $g(y) \neq g(y')$. Consequently, (c) of $(\mathcal{S}_{\ell+1})$ follows from (c') and the definition of $C_{\ell+1}(z)$ and $C_{\ell+1}(z')$.

We have therefore verified (a)–(c) of $(\mathcal{S}_{\ell+1})$, thus concluding the induction step. The proof of Lemma 18 follows by induction. □

5. Concluding remarks

Theorem 1 asserts the existence of a partition universal graph G for the class of graphs $\mathcal{H}_{\Delta,n}$ with G having $O(n^{2-1/\Delta}\log^{1/\Delta} n)$ edges. We believe it would be rather interesting to decide whether one can substantially improve on this upper bound. In particular, we believe that bringing this bound down to a bound of the form $O(n^{2-1/\Delta-\varepsilon})$ for some $\varepsilon > 0$ would require a completely new idea. The only lower bound that we know is of the form $\Omega(n^{2-2/\Delta})$ (see Remark 2(i)).

Our proof of Theorem 1 is heavily based on random graphs, and we do not know how to prove this result or anything numerically similar by constructive means. In particular, for instance, we do not know whether (N, d, λ)-graphs with reasonable parameters are partition universal for $\mathcal{H}_{\Delta,n}$.

Another interesting question is whether one can prove Theorem 1 without the regularity method.

References

[1] N. Alon, M. R. Capalbo, Y. Kohayakawa, V. Rödl, A. Ruciński, and E. Szemerédi, *Universality and tolerance* (*extended abstract*), Proceedings of the 41st IEEE Annual Symposium on Foundations of Computer Science (FOCS 2000), 2000, pp. 14–21. 2, 2

[2] ———, *Near-optimum universal graphs for graphs with bounded degrees* (*extended abstract*), Approximation, Randomization and Combinatorial Optimization: Algorithms and Techniques (M. Goemans and K. J. abd J.D.P. Rolim abd L. Trevisan, eds.), Lecture Notes in Computer Science, 2129, Springer-Verlag, 2001, pp. 170–180. 2

[3] N. Alon and M. R. Capalbo, *Sparse universal graphs for bounded-degree graphs*, Random Structures Algorithms **31** (2007), no. 2, 123–133. 2

[4] ———, *Optimal universal graphs with deterministic embedding*, Proceedings of the 19th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA 2008), 2008, pp. 373–378. 2

[5] N. Alon and Z. Füredi, *Spanning subgraphs of random graphs*, Graphs Combin. **8** (1992), no. 1, 91–94. 4.2, 4.2

[6] J. Beck, *On size Ramsey number of paths, trees, and circuits. I*, J. Graph Theory **7** (1983), no. 1, 115–129. 1

[7] ———, *On size Ramsey number of paths, trees and circuits. II*, Mathematics of Ramsey theory, Algorithms Combin., vol. 5, Springer, Berlin, 1990, pp. 34–45. 1

[8] E. A. Bender and E. R. Canfield, *The asymptotic number of labeled graphs with given degree sequences*, J. Combinatorial Theory Ser. A **24** (1978), no. 3, 296–307. 2

[9] B. Bollobás, *Random graphs*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1985. 1

[10] V. Chvátal, V. Rödl, E. Szemerédi, and W. T. Trotter, Jr., *The Ramsey number of a graph with bounded maximum degree*, J. Combin. Theory Ser. B **34** (1983), no. 3, 239–243. 4.2

[11] R. A. Duke and V. Rödl, *On graphs with small subgraphs of large chromatic number*, Graphs Combin. **1** (1985), no. 1, 91–96. 2.1

[12] P. Erdős, *Problems and results in graph theory*, The theory and applications of graphs (Kalamazoo, Mich., 1980), Wiley, New York, 1981, pp. 331–341. 1

[13] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *The size Ramsey number*, Period. Math. Hungar. **9** (1978), no. 1-2, 145–161. 1

[14] J. Friedman and N. Pippenger, *Expanding graphs contain all small trees*, Combinatorica **7** (1987), no. 1, 71–76. 1

[15] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math. **34** (1978), 275–291 (1979). 1

[16] S. Gerke, Y. Kohayakawa, V. Rödl, and A. Steger, *Small subsets inherit sparse ε-regularity*, J. Combin. Theory Ser. B **97** (2007), no. 1, 34–56. 1, 1, 2.1, 2.1, 7, 8

[17] S. Gerke and A. Steger, *The sparse regularity lemma and its applications*, Surveys in combinatorics 2005 (University of Durham, 2005) (B. S. Webb, ed.), London Math. Soc. Lecture Note Ser., vol. 327, Cambridge Univ. Press, Cambridge, 2005, pp. 227–258. 1

[18] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerdi theorem*, 2004, 44pp. 1

[19] R. L. Graham and V. Rödl, *Numbers in Ramsey theory*, Surveys in combinatorics 1987 (New Cross, 1987), Cambridge Univ. Press, Cambridge, 1987, pp. 111–153. 1

[20] P. E. Haxell and Y. Kohayakawa, *The size-Ramsey number of trees*, Israel J. Math. **89** (1995), no. 1-3, 261–274. 1

[21] P. E. Haxell, Y. Kohayakawa, and T. Łuczak, *The induced size-Ramsey number of cycles*, Combin. Probab. Comput. **4** (1995), no. 3, 217–239. 1

[22] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. 1, 1

[23] Y. Kohayakawa, *Szemerédi's regularity lemma for sparse graphs*, Foundations of computational mathematics (Rio de Janeiro, 1997), Springer, Berlin, 1997, pp. 216–230. 1, 2

[24] Y. Kohayakawa and V. Rödl, *Regular pairs in sparse random graphs. I*, Random Structures Algorithms **22** (2003), no. 4, 359–434. 1, 2.1, 3.3

[25] ———, *Szemerédi's regularity lemma and quasi-randomness*, Recent advances in algorithms and combinatorics, CMS Books Math./Ouvrages Math. SMC, vol. 11, Springer, New York, 2003, pp. 289–351. 1, 2

[26] J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), János Bolyai Math. Soc., Budapest, 1996, pp. 295–352. 1

[27] J. Komlós, G. N. Sárközy, and E. Szemerédi, *Blow-up lemma*, Combinatorica **17** (1997), no. 1, 109–123. 1

[28] ———, *An algorithmic version of the blow-up lemma*, Random Structures and Algorithms **12** (1998), no. 3, 297–312. 1

[29] J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi, *The regularity lemma and its applications in graph theory*, Theoretical aspects of computer science (Tehran, 2000), Lecture Notes in Comput. Sci., vol. 2292, Springer, Berlin, 2002, pp. 84–112. 1

[30] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular k-uniform hypergraphs*, Random Structures Algorithms **28** (2006), no. 2, 113–179. 1

[31] V. Rödl, B. Nagle, J. Skokan, M. Schacht, and Y. Kohayakawa, *The hypergraph regularity method and its applications*, Proc. Natl. Acad. Sci. USA **102** (2005), no. 23, 8109–8113. 1

[32] V. Rödl and A. Ruciński, *Perfect matchings in ε-regular graphs and the blow-up lemma*, Combinatorica **19** (1999), no. 3, 437–452. 4.2, 4.2

[33] V. Rödl and E. Szemerédi, *On size Ramsey numbers of graphs with bounded degree*, Combinatorica **20** (2000), no. 2, 257–262. 1

[34] V. Rödl and J. Skokan, *Regularity lemma for k-uniform hypergraphs*, Random Structures Algorithms **25** (2004), no. 1, 1–42. 1

[35] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik. 1

[36] ———, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401. 2

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