

# OBFUSCATED DRAWINGS OF PLANAR GRAPHS

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ABSTRACT. Given a planar graph  $G$ , we consider drawings of  $G$  in the plane where edges are represented by straight line segments (which possibly intersect). Such a drawing is specified by an injective embedding  $\pi$  of the vertex set of  $G$  into the plane. Let  $fix(G, \pi)$  be the maximum integer  $k$  such that there exists a crossing-free redrawing  $\pi'$  of  $G$  which keeps  $k$  vertices fixed, i.e., there exist  $k$  vertices  $v_1, \dots, v_k$  of  $G$  such that  $\pi(v_i) = \pi'(v_i)$  for  $i = 1, \dots, k$ . We give examples of planar graphs  $G$  along with a drawing  $\pi$  for which  $fix(G, \pi) = O(\sqrt{n})$ . In fact, such a drawing  $\pi$  exists even if it is presupposed that the vertices occupy any prescribed set of points on the boundary of a convex body.

We also consider the parameter  $obf(G)$  of a graph  $G$  which is equal to the maximum number of edge crossings over all straight line drawings of  $G$ . We give examples of planar graphs with  $obf(G) \geq (\frac{9}{4} - o(1))n^2$  and prove that  $obf(T) \geq (\frac{13}{8} - o(1))n^2$  for every triangulation  $T$ . We also show that a given triangulation  $T$  can be efficiently drawn with at least  $0.69 obf(T)$  crossings.

## 1. INTRODUCTION

In a *plane graph*, each vertex  $v$  is a point in  $\mathbb{R}^2$  and each edge  $uv$  is represented as a continuous plane curve with endpoints  $u$  and  $v$ . All such curves are supposed to be non-self-crossing and any two of them either have no common point or share a common endvertex. An underlying abstract graph of a plane graph is called *planar*. A planar graph can be drawn as plane graph in many ways and the Wagner-Fáry-Stein theorem (see, e.g., [8]) states that there always exists a *straight line drawing* in which every edge is represented by a straight line segment.

Let  $V(G)$  denote the vertex set of a planar graph  $G$ . In this paper, by a *drawing* of  $G$  we mean an arbitrary injective map  $\pi : V(G) \rightarrow \mathbb{R}^2$ . This definition abstracts away edge drawing issues. Given a drawing  $\pi$ , we just suppose that each edge  $uv$  of  $G$  is drawn as the straight line segment with endpoints  $\pi(u)$  and  $\pi(v)$ . Due to possible edge crossings and even overlaps,  $\pi$  may not be a plane drawing of  $G$ . Hence it is natural to ask:

How many vertices have to be moved to obtain from  $\pi$  a plane (i.e., crossing-free) straight line drawing of  $G$ ?

Alternatively, we could allow in  $\pi$  curved edges without their exact specification; such a drawing could be always assumed to be a plane graph. Then our task would be to *straighten*  $\pi$  rather than eliminate edge crossings.

More formally, for a planar graph  $G$  and a drawing  $\pi$ , let

$$fix(G, \pi) = \max_{\pi'} |\{v \in V(G) : \pi'(v) = \pi(v)\}|$$

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where the maximum is taken over all plane straight line drawings  $\pi'$  of  $G$ . Furthermore, let

$$fix(G) = \min_{\pi} fix(G, \pi). \quad (1)$$

In other words,  $fix(G)$  is the maximum number of vertices which can be fixed in any drawing of  $G$  while “untangling” it.

Improving a recent result of Spillner and Wolff [11], Bose et al. [4] showed that

$$fix(G) \geq (n/3)^{1/4}$$

for every planar graph  $G$ , where here and in the rest of this paper  $n$  denotes the number of vertices of the graph under consideration. Better bounds on  $fix(G)$  are known for cycles [9], trees [5] and, more generally, outerplanar graphs [11]. In all these cases it was shown that  $fix(G) = \Omega(n^{1/2})$ . In this paper we are interested in upper bounds on  $fix(G)$ .

The *cycle* (resp. *path*; *empty graph*) on  $n$  vertices will be denoted by  $C_n$  (resp.  $P_n$ ;  $E_n$ ). Recall that the *join* of vertex-disjoint graphs  $G$  and  $H$  is the graph  $G * H$  consisting of the union of  $G$  and  $H$  and all edges between  $V(G)$  and  $V(H)$ . The graphs  $W_n = C_{n-1} * E_1$  (resp.  $F_n = P_{n-1} * E_1$ ;  $S_n = E_{n-1} * E_1$ ) are known as *wheels* (resp. *fans*; *stars*). By  $kG$  we denote the disjoint union of  $k$  copies of a graph  $G$ .

Several examples of (infinite sequences of) planar graphs with small  $fix(G)$  are known. Let  $fix^\circ(G)$  (resp.  $fix^-(G)$ ) be defined similarly to (1), where the minimum is taken over all  $\pi$  with the property that  $\pi(V(G))$  lies on a circle (resp. on a line). Pach and Tardos [9] showed that there are drawings of certain graphs, for which only  $o(n)$  vertices can stay fixed. In other word in order to untangle such a drawing, one has to shift almost all its vertices. In fact, this is already true for cycles. More precisely, Pach and Tardos [9] proved that

$$fix^\circ(C_n) = O((n \log n)^{2/3}). \quad (2)$$

Recently, Goaoc et al. [5] exhibited a sequence of triangulations  $T_n$  with  $fix^-(T_n) = O(\sqrt{n})$ . More specifically, they proved that

$$fix^-(P_n * P_2) < \sqrt{n} + 2. \quad (3)$$

In an early version of the present paper [7], that appeared shortly after and independently, we constructed 3-connected planar graphs  $H_n$  with  $fix^\circ(H_n) = o(n)$ . Though no explicit bound was specified in [7], a simple analysis of our construction reveals that

$$fix^\circ(H_n) \leq 2\sqrt{n} + 1 \quad (4)$$

(see also Theorem 2.2). In subsequent papers [11, 4] it was shown that a bound  $fix^-(G) = O(\sqrt{n})$  could be proved for  $G$  in special classes of planar graphs, as outerplanar and even acyclic graphs. Spillner and Wolff [11] showed

$$fix^-(F_n) < 2\sqrt{n} + 1 \quad (5)$$

and Bose et al. [4] established

$$fix^-(kS_k) \leq 3\sqrt{n} - 3 \quad (6)$$

In Section 2 we present our construction of graphs  $H_n$  and establish (4). We also reprove other known  $O(\sqrt{n})$ -bounds for  $fix(G)$  from a uniform perspective. A natural question arising here concerns the role of aligning the vertex set on a line or on a circle. Is the choice between a line or a circle essential in (3)–(6)? Roughly

speaking, our answer is no. More precisely, we show that there exists a drawing  $\pi$  with  $fix(H_n, \pi) = O(\sqrt{n})$  even if  $\pi(V(G))$  is presupposed to be any set of  $n$  points on the boundary of some arbitrary convex body (see Theorem 2.2). Note that the latter condition includes the both cases of aligning the vertices on a circle and on a line. The same statement holds true for the wheel and the fan graphs (see Theorem 2.5). Since  $F_n$  is a subgraph of  $P_n * P_2$ , we also obtain a similar strengthening, up to a constant factor, for (3).

Moreover, we show  $fix(kS_k, \pi) = O(\sqrt{n}2^{\alpha(\sqrt{n})})$ , where  $\alpha(\cdot)$  is the inverse Ackermann function, and  $\pi$  satisfies the same assumption as above (see Theorem 2.3). Maybe surprisingly, the proofs of Theorem 2.2 and Theorem 2.3 are similar (both rely on Lemma 2.1), although they deal with very different graphs.

We conclude Section 2 with proving

$$fix^-(G) = fix(G)$$

for all planar graphs  $G$  (see Theorem 2.6). The proof uses ideas from Bose et al. [4, Lemma 1]. It follows that

$$fix^-(G) \leq fix^\circ(G).$$

Consequently, upper bounds on  $fix(G, \pi)$  for drawings  $\pi$  with the condition that  $\pi(V(G))$  is an arbitrary set of points on the boundary of a convex plane body are formally stronger, because we do not know if  $fix^\circ(G) = O(fix(G))$ . In view of those results it might be interesting to consider the difference of  $fix^-(G)$  and  $fix^\circ(G)$  for arbitrary planar graphs. However, we shall not investigate this issue here.

In Section 3 we consider another parameter of straight line graph drawings with edge crossings, which was considered in [16]. We consider straight line graph drawings with no edge overlaps, i.e., any two segments representing edges can have at most one point in common. An *edge crossing* in such a drawing  $\pi$  is an unordered pair of edges having a common inner point. We denote the total number of edge crossings in  $\pi$  by  $obf(G, \pi)$  and define the *obfuscation complexity* of the graph  $G$  by

$$obf(G) = \max_{\pi} obf(G, \pi),$$

where the maximum runs over all straight line drawings of  $G$  with no edge overlaps. This definition needs not be restricted to planar graphs and  $obf(G)$  could be viewed as the antipode of the *rectilinear crossing number* (see, e.g., [15]). Along with  $fix(G)$ , the obfuscation complexity of a planar graph is motivated by the popular web puzzle called *Planarity Game* [12].

In [16] it was shown that

$$obf(G) < 3n^2$$

for all planar graphs  $G$ . It is unknown how tight this bound is. In this paper we construct an infinite family of planar graphs  $G$  along with their drawings showing that

$$obf^\circ(G) \geq \left(\frac{9}{4} - o(1)\right)n^2,$$

where  $obf^\circ(G)$  is the variant of  $obf(G)$  for drawings with all vertices lying on a circle (see Theorem 3.2). We also prove that  $obf(T) \geq (\frac{13}{8} - o(1))n^2$  for every triangulation  $T$  (see Corollary 3.4).

While the computational complexity of  $obf(G)$  is unclear, it was shown in [16] that this graph parameter can be approximated within a factor of  $1/3$ . More

precisely, in [16] it was proved, that given an arbitrary graph  $G$ , one can efficiently construct a drawing  $\pi$  with all vertices on a circle so that

$$obf(G, \pi) \geq \frac{1}{3} obf(G).$$

In this paper we improve this approximation result for the class of maximal planar graphs, i.e., triangulations. We show that every triangulation  $T$  can be efficiently drawn with at least  $c \cdot obf(T)$  crossings, for every  $c < 39/56$  (see Theorem 3.5).

## 2. GRAPHS WITH SMALL $fix(G)$

We will use Davenport-Schinzel sequences and introduce them below (see, e.g., [1] for more details). An integer sequence  $S = s_1 \dots s_n$  is called a  $(k, p)$ -Davenport-Schinzel sequence if the following conditions are met:

- $1 \leq s_i \leq k$  for each  $i \leq n$ ;
- $s_i \neq s_{i+1}$  for each  $i < n$ ;
- $S$  contains no subsequence  $xyxyxy \dots$  of length  $p + 2$  for any  $x \neq y$ .

By a *subsequence* of  $S$  we mean any sequence  $s_{i_1} s_{i_2} \dots s_{i_m}$  with  $i_1 < i_2 < \dots < i_m$ . The maximum length of a  $(k, p)$ -Davenport-Schinzel sequence will be denoted by  $\lambda_p(k)$ . We are interested in the particular case of  $p = 4$ .

We inductively define a family of functions over positive integers:

$$\begin{aligned} A_1(n) &= 2n & n &\geq 1, \\ A_k(1) &= 2 & k &\geq 1, \\ A_k(n) &= A_{k-1}(A_k(n-1)) & n &\geq 2, k \geq 2. \end{aligned}$$

*Ackermann's function* is defined by  $A(n) = A_n(n)$ . This function grows faster than any primitive recursive function. The inverse of Ackermann's function is defined by  $\alpha(n) = \min \{t \geq 1 : A(t) \geq n\}$ .

Agarwal, Sharir, and Shor [2] proved that  $\lambda_4(k) = O(k2^{\alpha(k)})$ . Note that  $\alpha(n)$  grows very slowly, e.g.,  $\alpha(n) \leq 4$  for all  $n$  up to  $A(4)$ , which is the exponential tower of twos of height 65536. Thus, the bound for  $\lambda_4(k)$  is nearly linear in  $k$ .

Sometimes it will be convenient to identify a sequence  $S = s_1 \dots s_n$  with all its cyclic shifts. This way  $s_j s_n s_1 s_i$ , where  $i < j$ , is a subsequence of  $S$ . In such circumstances we will call a sequence *circular*. Subsequences of  $S$  will be considered also circular sequences. Note that the set of all circular subsequences is the same for  $S$  and any its shift. The length of  $S$  will be denoted by  $|S|$ .

**Lemma 2.1.** *Let  $k, s \geq 1$  and  $S^{k,s}$  be the circular sequence consisting of  $s$  successive blocks of the form  $12 \dots k$ .*

1. *Suppose that  $S$  is a subsequence of  $S^{k,s}$  with no 4-subsubsequence of the form  $xyxy$ , where  $x \neq y$ . Then  $|S| < k + s$ .*
2. *Suppose that  $S$  is a subsequence of  $S^{k,s}$  with no 6-subsubsequence of the form  $xyxyxy$ , where  $x \neq y$ . Then  $|S| < \lambda_4(k) + s \leq O(k2^{\alpha(k)}) + s$ .*

*Proof.* **1.** We proceed by the double induction on  $k$  and  $s$ . The base case where  $k = 1$  and  $s$  is arbitrary is trivial. Let  $k \geq 2$  and consider an  $S$  with no forbidden subsequence. If every of the  $k$  elements occurs in  $S$  at most once, then  $|S| \leq k$  and the claimed bound is true. Otherwise, without loss of generality we suppose that  $S$  contains  $\ell \geq 2$  occurrences of  $k$ . Let  $A_1, \dots, A_\ell$  (resp.  $B_1, \dots, B_\ell$ ) denote the parts of  $S$  (resp.  $S^{k,s}$ ) between these  $\ell$  elements. Thus,  $|S| = \ell + \sum_{i=1}^{\ell} |A_i|$ .

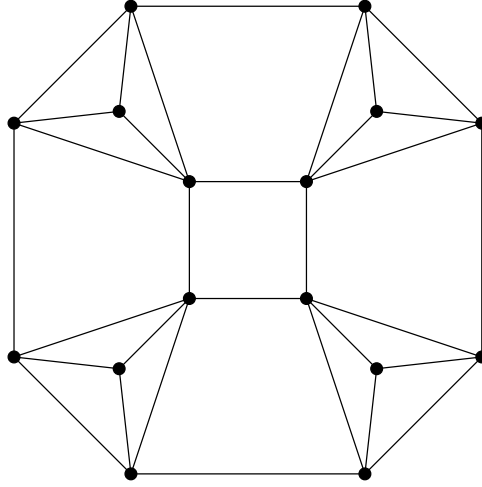


FIGURE 1. An example of  $H$  for  $k = 4$ .

Denote the number of elements with at least one occurrence in  $A_i$  by  $k_i$ . Each element  $x$  occurs in at most one of the  $A_i$ 's because otherwise  $S$  would contain a subsequence  $xkxk$ . It follows that  $\sum_{i=1}^{\ell} k_i \leq k-1$ . Note that, if we append  $B_i$  with an element  $k$ , it will consist of blocks  $12 \dots k$ . Denote the number of these blocks by  $s_i$  and notice the equality  $\sum_{i=1}^{\ell} s_i = s$ . Since  $A_i$  has no forbidden subsequence, we have  $|A_i| \leq k_i + s_i - 1$ . If  $k_i \geq 1$ , this follows from the induction assumption because  $A_i$  can be regarded as a subsequence of  $S^{k_i, s_i}$ . If  $k_i = 0$ , this is also true because then  $|A_i| = 0$ . Summarizing, we obtain  $|S| \leq \ell + \sum_{i=1}^{\ell} (k_i + s_i - 1) \leq \ell + (k-1) + s - \ell < k + s$ .

2. Let  $S'$  be obtained from  $S$  by shrinking each block  $z \dots z$  of the same elements to  $z$ . Since  $S'$  is a  $(k, 4)$ -Davenport-Schinzel sequence, we have  $|S'| \leq \lambda_4(k)$ . Note now that any two elements neighboring in a shrunken block are at the distance at least  $k-1$  in  $S^{k, s}$ . It easily follows that the total number of elements deleted in  $S$  is less than  $s$ .  $\square$

If  $W \subseteq V(G)$ , then  $G[W]$  denotes the subgraph induced by  $G$  on  $W$ .

**Theorem 2.2.** *Let  $k \geq 3$ ,  $n = k^2$ , and  $H$  be a 3-connected plane graph with  $n$  vertices having the following property: Its vertex set can be split into  $k$  equal parts  $V(H) = V_1 \cup \dots \cup V_k$  so that each  $H[V_i]$  is a triangulation and these  $k$  triangulations lie in the outer faces of each other (see Fig. 1). Let  $X$  be an arbitrary set of  $n$  points on the boundary  $\Gamma$  of a convex plane body. Then there is a drawing  $\pi : V(H) \rightarrow X$  such that  $\text{fix}(H, \pi) \leq 2\sqrt{n} + 1$ .*

*Proof.* Let  $X = \{x_1, \dots, x_n\}$ , where the points in  $X$  are numbered in the order of their appearance along  $\Gamma$ . Fix  $\pi$  to be an arbitrary map such that  $\pi(V_i) = \{x_i, x_{i+k}, x_{i+2k}, \dots, x_{i+(k-1)k}\}$  for each  $i \leq k$ .

Let  $\pi'$  be an arbitrary crossing-free straight line redrawing of  $H$ . We have to show that not many vertices of  $H$  keep the same location in  $\pi'$  as they had in  $\pi$ . Denote  $A_i = \{\pi(v) : v \in V_i, \pi(v) = \pi'(v)\}$  and  $A = \bigcup_{i=1}^k A_i$ . The union  $A$

consists of exactly those vertices that keep their position under transition from  $\pi$  to  $\pi'$ . Thus, we have to bound the number of vertices in  $A$  from above.

Denote the complement of the outer face of  $H[V_i]$  in  $\pi'$  by  $T_i$ . Since  $H[V_i]$  is a triangulation,  $T_i$  is a triangle containing all  $\pi'(V_i)$ . Recall that  $H$  is 3-connected. By Whitney's theorem (see, e.g., [6]),  $\pi'$  is equivalent to the original plane embedding of  $H$ , which we denote by  $\delta$ . This means that one of the following two cases occurs:

- A:  $\pi'$  is obtainable from  $\delta$  by a plane homeomorphism.
- B:  $\pi'$  is obtainable by a plane homeomorphism from  $\delta_F$ , where  $F$  is an inner face of  $\delta$  and  $\delta_F$  is an embedding of  $H$  obtained from  $\delta$  by making the face  $F$  outer.

By construction, the regions occupied by the  $H[V_i]$ 's are pairwise disjoint in  $\delta$ . For  $\pi'$  this implies that, if we have Case A, then all  $k$  triangles  $T_i$  are pairwise disjoint. The same holds true in Case B if  $F$  is not a face of any  $H[V_i]$ -fragment. If  $F$  is a face of some  $H[V_j]$ -fragment, then the  $T_i$ 's are pairwise disjoint with one exception for the triangle  $T_j$ , which contains all the others.

Consider first the case that all the triangles are pairwise disjoint. Since  $A_i \subset T_i$ , the convex hulls of these sets of points are pairwise disjoint. Label each  $x_j$  by the index  $i$  for which  $x_j \in \pi(V_i)$  and consider the circular sequence of these labels in the order of their appearance along  $\Gamma$ . This is exactly the sequence  $S^{k,k}$  as in Lemma 2.1. Let  $S$  be the subsequence corresponding to the points in  $A$ . Since the points in  $A_i$  are labeled by  $i$ , we see that  $S$  has no subsequence of the form  $xyxy$ . By Lemma 2.1,  $|A| = |S| < 2k$ .

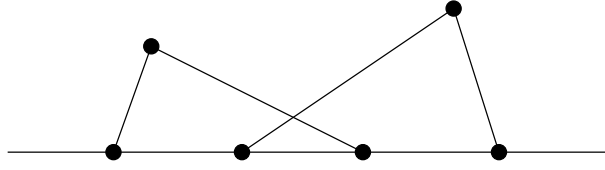
Consider now the case that the triangles  $T_i$  are pairwise disjoint with the exception, say, for  $T_k$ . Let  $a, b, c \in V_k$  be the vertices on the boundary of the outer face of  $H[V_k]$  in  $\delta$ . Let  $T$  denote the geometric triangle with vertices  $\pi'(a)$ ,  $\pi'(b)$ , and  $\pi'(c)$ . Note that  $A_i \subset T$  for all  $i < k$ . Note also that  $A_k \subset \Gamma \setminus T$ . The set  $\Gamma \setminus T$  consists of at most three continuous components; denote the corresponding parts of  $A_k$  by  $A'_k, A'_{k+1}, A'_{k+2}$ . Consider the circular sequence  $S$  as above with the following modification: the vertices in  $A'_{k+1}$  are relabeled with  $k+1$  and the vertices in  $A'_{k+2}$  are relabeled with  $k+2$  (the vertices in  $A'_k$  keep label  $k$ ). This modification rules out any  $xyxy$ -subsequence. Note that the modified  $S$  can be considered a subsequence of  $S^{k+2,k}$ . By Lemma 2.1, we have  $|A| \leq 2k+1$ .  $\square$

The following result is a variation of [4, Theorem 5]. The first part of Theorem 2.3 restates [4, Theorem 5] with a worse factor in front of  $\sqrt{n}$ ; we include it for an expository purpose. Somewhat surprisingly, the proof of this part is based on part 1 of Lemma 2.1, which we already used to prove Theorem 2.2. The second part, which is of our primary interest, requires a more delicate analysis based on part 2 of Lemma 2.1. Recall that  $S_k$  denotes the star with  $k-1$  leaves.

**Theorem 2.3.** *For all integer  $k \geq 2$  we have*

1.  $\text{fix}^-(kS_k) < 7\sqrt{n}$ ;
2. Let  $n = k^2$  and  $X$  be an arbitrary set of  $n$  points on the boundary  $\Gamma$  of a convex plane body  $P$ . Then there exists a drawing  $\pi$  of the forest  $kS_k$  such that  $\pi(V(kS_k)) = X$  and  $\text{fix}(kS_k, \pi) = O(\sqrt{n}2^{\alpha(\sqrt{n})})$ .

*Proof.* Denote  $V = V(kS_k)$ . Let  $V = \bigcup_{i=1}^k V_i \cup C$ , where each  $V_i$  consists of all  $k-1$  leaves in the same star component and  $C$  consists of all  $k$  central vertices.


 FIGURE 2. Proof of part 1 of Theorem 2.3: an *abab*-subsequence in  $A$ .

1. Fix a line  $\ell$  and pick up points  $x_1, \dots, x_n$  lying on  $\ell$  in this order. Consider a drawing  $\pi : V \rightarrow \ell$  such that

$$\begin{aligned} \pi(V_i) &= \{x_i, x_{i+k}, x_{i+2k}, \dots, x_{i+(k-2)k}\} \text{ for each } i \leq k, \\ \pi(C) &= \{x_{n-k+1}, x_{n-k+2}, \dots, x_n\}. \end{aligned} \quad (7)$$

Let  $\pi'$  be a crossing-free straight line redrawing of  $kS_k$ . We have to estimate the number of fixed vertices, i.e., those vertices participating in  $F = \{\pi(v) : v \in V, \pi(v) = \pi'(v)\}$ . For this purpose we split  $F$  into four parts:  $F = A \cup B \cup D \cup E$  where  $A$  (resp.  $B$ ;  $D$ ) consists of the fixed leaves adjacent to central vertices located in  $\pi'$  above  $\ell$  (resp. below  $\ell$ ; on  $\ell$ ) and  $E$  consists of the fixed central vertices.

Trivially,  $|E| \leq k$  and it is easy to see that  $|D| \leq 2|E|$ . Let us estimate  $|A|$  and  $|B|$ . Exactly as in the proof of Theorem 2.2, we label each  $x_j$  by the index  $i$  for which  $x_j \in \pi(V_i)$  and view  $x_1x_2 \dots x_{n-k}$  as the circular sequence  $S^{k,k-1}$  defined in Lemma 2.1. Let  $S$  be the subsequence induced by the points in  $A$ . Note that  $S$  does not contain any subsequence *abab* because otherwise we would have an edge crossing in  $\pi'$  (see Fig. 2). By part 1 of Lemma 2.1, we have  $|A| = |S| < 2k$ . The same applies to  $B$ . It follows that  $|F| = |A| + |B| + |D| + |E| < 7k$ , as claimed.

2. It is known that the boundary of a convex plane body is a rectifiable curve and, therefore, we can speak of the length of  $\Gamma$  or its arcs. Clearly, the convex body  $P$  plays a nominal role and can be varied once  $X$  is fixed. Thus, to avoid unnecessary technical complications in the forthcoming argument, without loss of generality we can suppose that the boundary curve  $\Gamma$  contains a finite number of maximal straight line segments.

We will use the following terminology. A *chord* is a straight line segment whose endpoints lie on  $\Gamma$ . An *arrow* is a directed chord with one endpoint called *head* and the other called *tail*. Call an arrow a *median* if its endpoints split  $\Gamma$  into arcs of equal length. Fix the ‘‘clockwise’’ order of motion along  $\Gamma$  and color each non-median arrow in one of two colors, red if the shortest way along  $\Gamma$  from the tail to the head is clockwise and blue if it is counter-clockwise.

Given a point  $a$  outside  $P$ , we define *quiver*  $Q_a$  as follows. For each line going through  $a$  and intersecting  $\Gamma$  in exactly two points,  $h$  and  $t$ , the  $Q_a$  contains the arrow  $th$  directed so that the head is closer to  $a$  than the tail.

Given a non-median arrow  $th$ , we will denote the shorter component of  $\Gamma \setminus \{t, h\}$  by  $\Gamma[t, h]$ . Our argument will be based on the following elementary fact.

*Claim A.* Let arrows  $th$  and  $t'h'$  be in the same quiver  $Q$  and have the same color. Suppose that  $\Gamma[t', h']$  is shorter than  $\Gamma[t, h]$ . Then both  $t'$  and  $h'$  lie in  $\Gamma[t, h]$ .

*Proof of Claim A.* Let  $t^*h^*$  be the median in  $Q$ . Since  $th$  and  $t'h'$  are of the same color, the four points  $t, h, t', h'$  are in the same component of  $\Gamma \setminus \{t^*, h^*\}$ . The claim easily follows from the fact that the chords  $th$  and  $t'h'$  do not cross (see Fig. 3).  $\triangleleft$

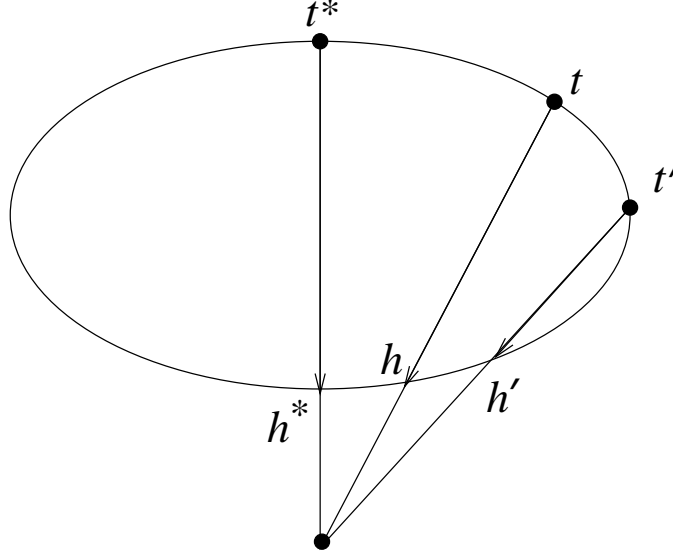


FIGURE 3. Proof of Claim A.

After these preliminaries, we begin with the proof. Let  $x_1, \dots, x_n$  be a listing of points in  $X$  along  $\Gamma$ . Fix  $\pi$  to be an arbitrary map satisfying (7). Let  $\pi'$  be a crossing-free redrawing of  $kS_k$ . Perturbing  $\pi'$  a little (and using the regularity assumption made about  $\Gamma$ ), we can ensure that every edge in  $\pi'$  with one endpoint  $\pi'(v)$  on  $\Gamma$  and the other endpoint outside satisfies the following two conditions:

- (1) The edge intersects  $\Gamma$  in at most two points, including  $\pi'(v)$  (this is automatically true if  $\Gamma$  contains no straight line segment).
- (2) If the edge intersects  $\Gamma$  in two points, it splits  $\Gamma$  into components having different lengths.

Assume that  $\pi'$  meets these conditions. Let  $v$  be a leaf adjacent to a central vertex  $c$ . Suppose that  $\pi'(v) \in \Gamma$ ,  $\pi'(c) \notin P$ , and the segment  $\pi'(v)\pi'(c)$  crosses  $\Gamma$  at a point  $h \neq \pi'(v)$ . By Condition 2, the arrow  $\pi'(v)h$  is not a median and hence colored in red or blue. We color each such  $\pi'(v)$  in red or blue correspondingly.

Now we split the set of fixed vertices  $F$  into five parts. Let  $E$  consist of the fixed central vertices,  $I$  (resp.  $O$ ) consist of those fixed leaves such that the edges emanating from them are completely inside (resp. outside)  $P$ , and  $R$  (resp.  $B$ ) consist of the red (resp. blue) fixed leaves. By Condition 1, we have  $F = E \cup I \cup O \cup R \cup B$ .

Trivially,  $|E| \leq k$ . Similarly to the proof of the first part of the theorem, notice that the subsequences of  $S^{k,k-1}$  corresponding to  $I$  and  $O$  do not contain *abab*-subsequences. By part 1 of Lemma 2.1, we have  $|I| < 2k$  and  $|O| < 2k$ .

Finally, consider the subsequence  $S$  of  $S^{k,k-1}$  corresponding to  $R$  and show that it does not contain any *ababab*-subsequence. Assume, to the contrary, that such a subsequence exists. This means that  $x_1 \dots x_{n-k}$  contains two interchanging subsequences  $a_1 a_2 a_3$  and  $b_1 b_2 b_3$  whose elements belong to two different star components of  $\pi'$ , with central vertices  $a$  and  $b$  respectively. Since  $a_1, a_2, a_3$  are red,

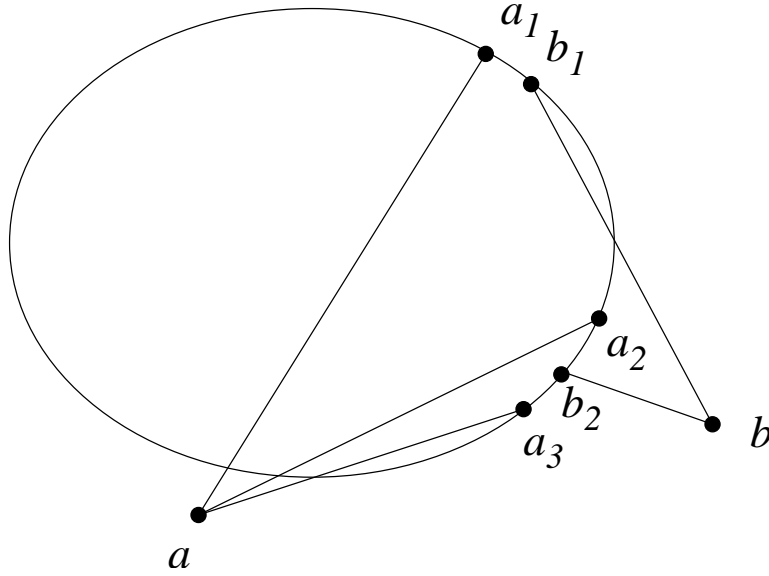


FIGURE 4. Proof of part 2 of Theorem 2.3:  $ababab$ -subsequence in  $R$ .

Claim A implies that, say,  $a_2$  and  $a_3$  lie on the shorter arc of  $\Gamma$  cut off by the edge  $aa_1$  (see Fig. 4).

Without loss of generality, let  $b_1$  be between  $a_1$  and  $a_2$  and  $b_2$  be between  $a_2$  and  $a_3$ . Since  $b_1$  and  $b_2$  are red and  $\pi'$  is crossing-free, it must be the case that  $bb_1$  intersects  $\Gamma[a_1, a_2]$  and  $bb_2$  intersects  $\Gamma[a_2, a_3]$  (in another point). This makes a contradiction with Claim A.

Thus,  $S$  is  $ababab$ -free and, by part 2 of Lemma 2.1, we have  $|R| = |S| \leq k2^{O(\alpha(k))}$ . All the same applies to  $B$ . Summarizing, we see that  $|F| = |E| + |I| + |O| + |R| + |B| \leq k2^{O(\alpha(k))}$ , as claimed.  $\square$

An undirected circular sequence  $a_1a_2 \dots a_n$  is not distinguished from any cyclic shift of it or of its reversal  $a_n \dots a_2a_1$ . Two number sequences  $A = a_1a_2 \dots a_n$  and  $B = b_1b_2 \dots b_n$  are called *similar* if  $a_i < a_j$  whenever  $b_i < b_j$  and vice versa. If  $A$  and  $B$  are similar or become similar after cyclic shift and/or reversal of one of them, then they are considered similar as undirected circular sequence. Suppose that all numbers in a sequence are pairwise distinct. Note that then there is only one similarity type of an undirected circular sequence of length 3. Furthermore, there are exactly three similarity types of length 4, namely 1234, 1243, and 1324.

**Lemma 2.4.**

1. Any 4-subsequence of the undirected circular sequence  $123 \dots \ell$  is similar to 1234.
2. Given  $k \geq 2$  and  $2k + 1 \leq \ell \leq 2k^2$ , let  $E^{k,\ell}$  be the following undirected circular sequence of length  $\ell$ :

$$k, k - 1, \dots, 1, 2k, 2k - 1, \dots, k + 1, 3k, 3k - 1, \dots .$$

Let  $S$  be a subsequence of  $E^{k,\ell}$ .

- (a) Suppose that all 4-subsequences of  $S$  are similar to 1234. Then  $|S| \leq 2k$ .
- (b) Suppose that  $S$  does not contain any subsequence similar to 1324. Then  $|S| \leq 4k - 2$ .

*Proof.* Part 1 is easy to check. Consider part 2. Note that  $E^{k,\ell}$  consists of at most  $2k$  (and at least 3) monotone blocks of length at most  $k$  each. Assume that  $|S| > 2k$ . As easily seen,  $S$  must have elements from at least 3 monotone blocks of  $E^{k,\ell}$ . Moreover,  $S$  must have at least 2 elements from some block. Thus,  $S$  has elements  $a_1, a_2, b, c$ , where  $a_1, a_2$  occur in the same monotone block of  $E^{k,\ell}$ , while  $b$  and  $c$  occur in two other blocks. Direct verification shows that such 4 elements always induce a subsequence of  $S$  similar either to 1243 or to 1324. This proves claim (a).

To prove claim (b), assume that  $|S| \geq 4k - 1$ . It is easy to see that there are at least 3 monotone blocks such that  $S$  has at least 2 elements in each of them. Let  $A, B$ , and  $C$  be such blocks listed in the order as they appear in  $E^{k,\ell}$ . Pick up four elements  $a \in A, b_1, b_2 \in B$ , and  $c \in C$  participating in  $S$ . As easily seen, elements  $a, b_1, b_2, c$  induce a subsequence similar to 1324.  $\square$

The following result is a variation of [5, Theorem 7] and [11, Theorem 4].

**Theorem 2.5.** *Let  $n \geq 4$  and  $X$  be an arbitrary set of  $n$  points on the boundary  $\Gamma$  of a convex plane body  $P$ . Then the following claims are true.*

1. *There is a drawing  $\pi$  of the wheel  $W_n$  such that  $\pi(V(W_n)) = X$  and  $\text{fix}(W_n, \pi) < 2\sqrt{2n} + 5$ .*
2. *There is a drawing  $\pi$  of the fan  $F_n$  such that  $\pi(V(F_n)) = X$  and  $\text{fix}(F_n, \pi) < 4\sqrt{2n} + 4$ .*

*Proof.* **1.** Let  $V(W_n) = \{v_0, v_1, \dots, v_{n-1}\}$ , where  $v_0$  is the central vertex and  $v_1, \dots, v_{n-1}$  induce a cycle and appear in it in this order. The wheel  $W_n$  is a 3-connected graph. The Whitney theorem implies that, in every crossing-free drawing of  $W_n$ , the vertices  $v_1, \dots, v_{n-1}$  appear around  $v_0$  exactly in this undirected circular sequence.

Let  $x_0, \dots, x_{n-1}$  be a listing of points of  $X$  as they go along  $\Gamma$ . We set  $\pi(v_0) = x_0$ , though this choice does not play any important role. Now we have to define  $\pi$  on the other vertices so that their order along  $\Gamma$  will disagree with the canonical Whitney order as much as possible. For this purpose, we place vertices  $v_1, \dots, v_{n-1}$  on points  $x_1, \dots, x_{n-1}$  so that the indices of the  $v_i$ 's along  $\Gamma$  form sequence  $E^{k,n-1}$  as in part 2 of Lemma 2.4, where  $k$  is determined from the condition  $2(k-1)^2 < n-1 \leq 2k^2$ .

Let  $\pi'$  be a crossing-free straight line redrawing of  $W_n$ . Similarly to the proof of part 2 of Theorem 2.3, we can suppose that every edge emanating from  $\pi'(v_0)$  has at most two common points with  $\Gamma$ . We will show that the set  $A = \{\pi(v) : v \in V(W_n) \setminus \{v_0\}, \pi(v) = \pi'(v)\}$  contains less than  $4k + 2$  points. We split  $A$  into two parts,  $A_0$  and  $A_1$ , where  $A_0$  (resp.  $A_1$ ) consists of those  $\pi(v) = \pi'(v)$  for which the edge  $\pi'(v_0)\pi'(v)$  lies completely outside  $P$  (resp. at least partially inside  $P$ ), see Fig. 5. If  $|A| \geq 4k + 2$ , then  $|A_0| \geq 2k + 1$  or  $|A_1| \geq 2k + 1$ . As the two cases are similar, we consider the latter.

Let  $S$  be the subsequence of  $E^{k,n-1}$  induced by the points in  $A_1$ . It follows from the convexity of  $P$  that  $S$  is a sequence of appearances of (the indices of) some vertices around the central vertex in  $\pi'$ . By part 1 of Lemma 2.4, every 4-subsequence of  $S$  must be similar to the undirected circular sequence 1234. On the

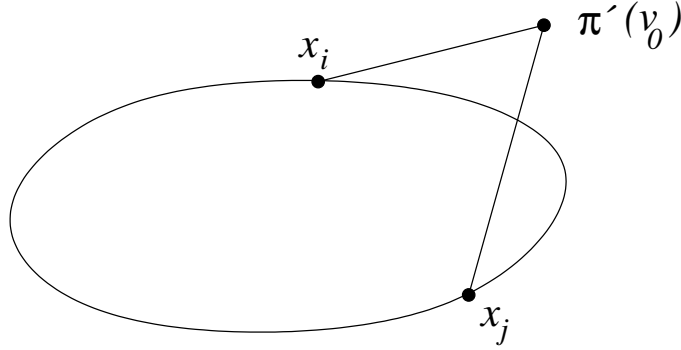


FIGURE 5. Definition of  $A_0$  and  $A_1$  in the case of  $\pi'(v_0) \notin P$ :  
 $x_i \in A_0$ ,  $x_j \in A_1$ .

other hand, from the condition  $|S| \geq 2k+1$  and part 2(a) of Lemma 2.4 we conclude that  $S$  contains a 4-subsequence which is not similar to 1234. This contradiction proves that  $|A| \leq 4k+1$ , giving us the required bound for  $fix(W_n, \pi)$ .

2. Similarly to the first part, let  $V(F_n) = \{v_0, v_1, \dots, v_{n-1}\}$  where  $v_0$  is the central vertex and  $v_1 \dots v_{n-1}$  is the path of the remaining vertices. Analysis of  $F_n$  is more complicated because this graph is not 3-connected and has a number of essentially different plane drawings. Let  $\alpha$  be such a drawing of  $F_n$ . Label each edge  $\alpha(v_0)\alpha(v_i)$  with number  $i$  and denote the sequence in which the labels follow each other around  $\alpha(v_0)$  by  $R_\alpha$ .

Our nearest goal is to show that  $R_\alpha$  has rather simple structure. For this purpose we consider two pieces of  $R_\alpha$ . Let  $R'_\alpha$  be the sequence of labels starting with 1, ending with  $n-1$ , and containing all intermediate labels if we go around  $\alpha(v_0)$  clockwise. Let  $R''_\alpha$  be the counter-clockwise analog of  $R'_\alpha$ .

*Claim B.* Both  $R'_\alpha$  and  $R''_\alpha$  are monotone.

*Proof of Claim B.* We proceed by induction on  $n$ . The base case of  $n=3$  is obvious. Suppose that the claim is true for all plane drawings of  $F_n$  and consider an arbitrary plane drawing  $\alpha$  of  $F_{n+1}$ . Let  $\beta$  be obtained from  $\alpha$  by erasing  $\alpha(v_n)$  along with the incident edges. Obviously,  $\beta$  is a plane drawing of  $F_n$ .

Let  $T$  be the triangle with vertices  $\alpha(v_0)$ ,  $\alpha(v_{n-1})$ , and  $\alpha(v_n)$ . Clearly, all points  $\alpha(v_i)$  for  $i \leq n-2$  are inside  $T$  or all of them are outside. In both cases,  $n-1$  and  $n$  are neighbors in  $R_\alpha$ . Therefore,  $R_\alpha$  is obtainable from  $R_\beta$  by inserting  $n$  on the one or the other side nearby  $n-1$ . It follows that  $R'_\alpha$  is obtained from  $R'_\beta$  either by appending  $n$  after  $n-1$  or by replacing  $n-1$  with  $n$  (the same concerns  $R''_\alpha$  and  $R''_\beta$ ). It remains to note that both operations preserve monotonicity.  $\triangleleft$

We will use a simple consequence of Claim B: for every plane drawing  $\alpha$  of  $F_n$ , the sequence  $R_\alpha$  contains no subsequence similar to 1324. The rest of the proof goes similarly to part 1.

We define  $\pi$  exactly as in part 1. Again, let  $\pi'$  be a crossing-free redrawing of  $F_n$ . Furthermore, denote the set of vertices remaining fixed under the transition from  $\pi$  to  $\pi'$  by  $A$  and split it into two parts  $A_0$  and  $A_1$  exactly as in part 1. Let  $S$  be the subsequence of  $E^{k,n-1}$  corresponding to  $A_1$  and assume that  $|A_1| \geq 4k-1$ . By part 2(b) of Lemma 2.4,  $S$  contains a subsequence similar to 1324. We come

to a contradiction as  $S$  is a subsequence of  $R_{\pi'}$ . Thus,  $|A_1| \leq 4k - 2$ . Similarly,  $|A_0| \leq 4k - 2$  and hence  $|A| \leq 8k - 4 < 4\sqrt{2n} + 4$ .  $\square$

We conclude this section with a general relation between  $fix(G)$  and  $fix^-(G)$ .

**Theorem 2.6.** *For every planar graph  $G$  we have  $fix(G) = fix^-(G)$ .*

Theorem 2.6 can be deduced from [4, Lemma 1]. For the reader's convenience, we give a self-contained proof.

*Proof.* We have to prove that  $fix(G) \geq fix^-(G)$  (the other direction follows directly from the definition). That is, given an arbitrary drawing  $\pi : V(G) \rightarrow \mathbb{R}^2$ , we have to show that it can be untangled while keeping at least  $fix^-(G)$  vertices fixed.

Choose Cartesian coordinates in the plane so that  $\pi(V(G))$  is located between the lines  $y = 0$  and  $y = 1$ . Let  $p_x, p_y : \mathbb{R}^2 \rightarrow \mathbb{R}$  denote the projections onto the  $x$ -axis and the  $y$ -axis respectively. We also suppose that the axes are chosen so that the map  $\lambda = p_x \pi$  is injective. Let us view  $\lambda$  as a drawing of  $G$ , aligning all the vertices on the line  $y = 0$ . By definition, there is a plane drawing  $\lambda'$  such that the set of fixed vertices  $F = \{v \in V(G) : \lambda'(v) = \lambda(v)\}$  has cardinality at least  $|F| \geq fix^-(G)$ .

Given a set  $A \subset \mathbb{R}^2$  and a real  $\varepsilon > 0$ , let  $O_\varepsilon(A)$  denote the  $\varepsilon$ -neighborhood of  $A$ . For each pair of disjoint edges  $e, e'$  in  $\lambda'$ , there is an  $\varepsilon$  such that  $O_\varepsilon(e) \cap O_\varepsilon(e') = \emptyset$ . Since  $G$  is finite, we can assume that the latter is true with the same  $\varepsilon$  for all disjoint  $e, e'$ .

We now define a drawing  $\pi' : V(G) \rightarrow \mathbb{R}^2$  by setting

$$\pi'(v) = \begin{cases} (p_x \pi(v), \varepsilon p_y \pi(v)) & \text{if } v \in F, \\ \lambda'(v) & \text{otherwise.} \end{cases}$$

Note that  $\pi'(v) \in O_\varepsilon(\lambda'(v))$  for every  $v \in V(G)$ . Since  $\lambda'$  is crossing-free, so is  $\pi'$ .

Finally, define a linear transformation of the plane by  $a(x, y) = (x, \varepsilon^{-1}y)$  and consider  $\pi'' = a\pi'$ . Clearly,  $\pi''$  is a plane drawing of  $G$  and all vertices in  $F$  stay fixed under the transition from  $\pi$  to  $\pi''$ .  $\square$

### 3. GRAPHS WITH LARGE $obf(G)$

We reserve  $m$  to denote the number of edges in a graph under consideration. The number of edges emanating from a vertex  $v$  is called *degree* of this vertex and will be denoted by  $\deg v$ . For a graph  $G$ , we set

$$\sigma(G) = \sum_{v \in V(G)} \deg^2 v$$

and define

$$\varepsilon(G) = \binom{m}{2} - \sum_{v \in V(G)} \binom{\deg v}{2},$$

which is the number of unordered pairs of non-adjacent edges in  $G$ . Note that

$$\varepsilon(G) = \frac{1}{2}(m^2 + m - \sigma(G)). \quad (8)$$

In [16] it was observed that

$$\varepsilon(G)/3 \leq obf^\circ(G) \leq obf(G) \leq \varepsilon(G). \quad (9)$$

In some interesting cases we are able to make these relations more precise.

**Lemma 3.1.** *Let  $G$  be an arbitrary (not necessary planar) graph.*

1. *If  $G$  is 3-colorable, then  $\text{obf}^\circ(G) \geq \frac{1}{2}\varepsilon(G)$ .*
2. *Let  $G$  be 4-colorable and  $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$  be a partition of  $V(G)$  into independent sets. Denote the number of edges between  $V_i$  and  $V_j$  by  $e_{ij}$  and set  $s = e_{12}e_{34} + e_{13}e_{24} + e_{14}e_{23}$ . Then  $\text{obf}^\circ(G) \geq \frac{1}{2}\varepsilon(G) - \frac{1}{6}s$ .*

*Proof.* **1.** Let  $V(G) = V_1 \cup V_2 \cup V_3$  be the partition of  $V(G)$  corresponding to some 3-coloring. Split a circle into three arcs and put each  $V_i$  on its own arc so that the vertices come on the arc in a random order. Let  $ab$  and  $xy$  be disjoint edges of  $G$ . Without loss of generality, suppose that  $a, x \in V_1$ . Then  $b, y \notin V_1$ . Consider the event that  $ab$  and  $xy$  cross under the condition that the location of  $V_2$  and  $V_3$  is specified. Once we fix the positions of  $b$  and  $y$  on the circle, the vertices  $a, x$  necessarily fall into the same arc with endpoints  $b$  and  $y$ . Of two ways to relatively order  $a$  and  $x$ , exactly one gives a crossing. Hence, the probability of crossing is exactly  $1/2$ .

It follows that the expected number of crossing pairs is equal to  $\frac{1}{2}\varepsilon(G)$ . At least this number should be achieved by some ordering of vertices within  $V_1, V_2, V_3$ .

**2.** Similarly to above, we split a circle into four arcs, assign to each  $V_i$  its own arc at random, and put the vertices of  $V_i$  on the assigned arc in a random order. If we take in  $G$  any two disjoint edges  $ab$  and  $xy$ , then the probability that they cross is  $1/3$  if they do not have an endpoint in a common part  $V_i$  and  $1/2$  otherwise. It remains to notice that the expected number of crossings is equal to  $\frac{1}{2}(\varepsilon(G) - s) + \frac{1}{3}s = \frac{1}{2}\varepsilon(G) - \frac{1}{6}s$ .  $\square$

Part 1 of Lemma 3.1 reveals sufficient conditions for a planar graph to have large obfuscation complexity.

**Theorem 3.2.** *There are planar graphs with*

$$\text{obf}(G) \geq \left(\frac{9}{4} - o(1)\right)n^2. \quad (10)$$

*Moreover, appropriate drawings of  $G$  are efficiently constructible.*

*Proof.* Combining the bound from part 1 of Lemma 3.1 and (8), it is easy to see that we have (10) whenever a 3-colorable  $G$  has  $(3 + o(1))n$  edges and maximum degree  $o(n)$ . Consider a particular example. Start with the regular hexagonal infinite lattice in  $\mathbb{R}^2$ . We regard it as a 2-colorable 3-regular infinite graph with all faces hexagons. Triangulate each face by putting in it a new vertex of degree 6. This results in a 3-colorable 6-regular plane triangulation. Let  $G$  be the subgraph induced by a large disk. Its boundary has size  $b = O(\sqrt{n})$  where  $n$  is the number of vertices inside the disk. Therefore,  $G$  has at least  $\frac{1}{2} \times 6(n - b) = (3 - o(1))n$  edges. Of course, the maximum degree is 6.

This way, we have an explicit partition of  $V(G)$  into 3 independent sets: two parts of the original hexagonal lattice and the third part consisting of the additional vertices. This allows us to efficiently implement the randomized construction from the proof of part 1 of Lemma 3.1. It can be derandomized by the standard *method of conditional expectations* (e.g., [3, Chapter 15]).  $\square$

In the rest we focus on triangulations of the plane. Recall that in this case we have  $m = 3n - 6$ .

**Lemma 3.3.** *For any triangulation  $T$  we have*

$$\frac{1}{2}\varepsilon(G) - \frac{1}{8}n^2 < \text{obf}^\circ(T) \leq \text{obf}(T) \leq \frac{3}{4}\varepsilon(G) - \frac{3}{8}n^2 + O(n).$$

*Proof. Lower bound.* Since every planar graph is 4-colorable, we can use part 2 of Lemma 3.1. It is enough to prove that  $s < \frac{3}{4}n^2$ .

Let  $F$  denote the set of all faces of  $T$  and  $F = F_1 \cup F_2 \cup F_3 \cup F_4$  be its partition where  $F_i$  is the set of those faces not containing any vertex from  $V_i$ . Denote  $f = |F|$  and  $f_i = |F_i|$ . By Euler's formula,  $f = 2n - 4$ . Note that

$$\sum_{i=1}^4 f_i^2 \geq 4 \left( \frac{1}{4} \sum_{i=1}^4 f_i \right)^2 = \frac{1}{4}f^2.$$

Let  $\{i, j, k, l\} = \{1, 2, 3, 4\}$ . Counting the faces in  $F_k \cup F_l$  in two ways, we come to equality  $f_k + f_l = 2e_{ij}$ . From here we easily infer that  $s = \frac{1}{4}(f^2 - \sum f_i^2)$ . Therefore,  $s \leq \frac{3}{16}f^2 = \frac{3}{4}(n-2)^2$ , as needed.

*Upper bound.* We proceed similarly as in the proof of [16, Theorem 3]. Consider an arbitrary straight line drawing  $\tau$  of  $T$ . Suppose that edges  $e$  and  $e'$  cross in  $\tau$ . If  $e$  belongs to a facial triangle  $K$  and  $e'$  to a facial triangle  $K'$ , we will say that this crossing is *contributed* by the pair  $\{K, K'\}$ . We will estimate from above the number of such contributions.

Let  $P$  denote the set of all pairs  $\{K, K'\}$  where  $K$  and  $K'$  are different facial triangles of  $T$ . For  $i = 0, 1, 2$ , let  $P_i$  be the set of all  $\{K, K'\} \in P$  such that  $K$  and  $K'$  have exactly  $i$  common vertices. Denote  $p = |P|$  and  $p_i = |P_i|$ . We have

$$p = \binom{f}{2} = (2n-5)(n-2) \quad \text{and} \quad p_2 = m = 3n-6.$$

It is also not hard to see that

$$p_1 = \sum_{v \in V(T)} \left( \binom{\deg v}{2} - \deg v \right) = \frac{1}{2}\sigma(T) - 3m.$$

By (8), we obtain

$$p_1 = \frac{1}{2}m(m-5) - \varepsilon(G) = \frac{1}{2}(3n-6)(3n-11) - \varepsilon(G).$$

Finally,  $p_0 = p - p_1 - p_2$ .

Notice now that a pair in  $P_0$  contributes at most 6 crossings, a pair in  $P_1$  contributes at most 3, and a pair in  $P_2$  contributes at most 1. It follows that

$$\text{obf}(T, \tau) \leq \frac{1}{4}(6p_0 + 3p_1 + p_2).$$

The factor of  $\frac{1}{4}$  is needed here because every crossing is contributed by 4 triangle pairs. A direct calculation yields

$$\text{obf}(T, \tau) \leq \frac{3}{4}\varepsilon(G) - \frac{3}{8}n^2 + O(n).$$

The bound holds for  $\text{obf}(T)$  as  $\tau$  was chosen arbitrarily.  $\square$

**Corollary 3.4.** *Let  $T$  be a triangulation on  $n$  vertices.*

1. *If the maximum degree  $\Delta(T) = o(n)$ , then  $\text{obf}^\circ(T) \geq (\frac{17}{8} - o(1))n^2$ .*
2. *In general we have  $\text{obf}^\circ(T) \geq (\frac{13}{8} - o(1))n^2$ .*

*Proof.* We use the lower bound given by Lemma 3.3, which we have to combine with a lower bound for  $\varepsilon(T)$ . By (8), we have

$$\varepsilon(T) > \frac{9}{2}n^2 - \frac{1}{2}\sigma(T) - 33n.$$

Note that  $\sigma(T) \leq 2m\Delta(T)$ . If  $\Delta(T) = o(n)$ , this gives us  $\varepsilon(T) \geq (\frac{9}{2} - o(1))n^2$  and implies part 1.

If vertex degrees are unbounded, we have to estimate  $\sigma(T)$  from above in a different way. Let  $k \geq 12$ . Call the degree of a vertex  $v$  *large* if  $\deg v \geq k$ . West and Will [14] proved that the sum of all large degrees in a planar graph is smaller than  $(2 + \frac{12}{k-6})n$ . It follows that

$$\sigma(T) < \left(2 + \frac{144}{(k-6)^2}\right)n^2 + k^2n.$$

Taking, say,  $k = (1 + o(1))n^{1/3}$ , we conclude that  $\sigma(T) \leq (2 + o(1))n^2$ . This gives us

$$\varepsilon(T) \geq \left(\frac{7}{2} - o(1)\right)n^2 \tag{11}$$

and implies part 2.  $\square$

**Theorem 3.5.** *Let  $c < 39/56$ . Given a triangulation  $T$ , one can efficiently construct a straight line drawing  $\tau$  of  $T$  so that  $\text{obf}(T, \tau) \geq c \cdot \text{obf}(T)$ .*

*Proof.* Efficient approximability of  $\text{obf}(T)$  within any factor of  $c < 39/56$  follows from the relationship between  $\text{obf}(T)$  and  $\varepsilon(T)$ . Combining the upper bound in Lemma 3.3 with the simple bound  $\varepsilon(T) < \frac{1}{2}m^2 < \frac{9}{2}n^2$ , we obtain

$$\text{obf}(T) \leq \left(\frac{2}{3} + o(1)\right)\varepsilon(T). \tag{12}$$

On the other hand, if we combine the lower bound in Lemma 3.3 with (11), we arrive at a converse inequality

$$\text{obf}(T) \geq \left(\frac{13}{28} - o(1)\right)\varepsilon(T). \tag{13}$$

Moreover, we can construct a drawing  $\tau$  with  $\text{obf}(T, \tau)$  meeting the bound of (13) as described in the proof of part 2 of Lemma 3.1. This construction is based on a 4-coloring of  $T$ . For the latter there is a quadratic algorithm [10]. Though the construction is probabilistic, it is derandomizable by the method of conditional expectations.  $\square$

#### 4. FURTHER QUESTIONS

1. By a theorem of Tutte [13], every 4-connected planar graph is Hamiltonian. Combining this fact with Pach-Tardos' bound (2), we see that  $\text{fix}(G) = o(n)$  for all 4-connected planar  $G$ . For 2-connected planar graphs this universality result is no longer true as, for example,  $\text{fix}(K_{2,n}) \geq n$ . Is it true for all 3-connected planar graphs or, at least, for all triangulations? The best what is currently known for all 3-connected planar graphs seems to be a bound  $\text{fix}(G) \leq \frac{2}{3}n + \frac{1}{3}$  observed in [16].

2. By Theorem 2.6, we have  $\text{fix}^-(G) = \text{fix}(G) \leq \text{fix}^\circ(G)$ . The inequality can be strict: for example,  $\text{fix}^-(K_4) = 2$  while  $\text{fix}^\circ(K_4) = 3$ . Is it true that

$fix^\circ(G) = O(fix(G))$ ? Currently we cannot prove this even for graphs  $G = kS_k$ , cf. Theorem 2.3.

3. Let  $obf(n) = \max_G obf(G)$  where the maximum is taken over planar  $G$  with  $n$  vertices. By Theorem 3.2 and [16, Theorem 3] we know that

$$(9/4 - o(1))n^2 \leq obf(n) < 3n^2.$$

It would be interesting to close this gap.

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