# Time Petri Nets: Theory, Tools and Applications

## Part II


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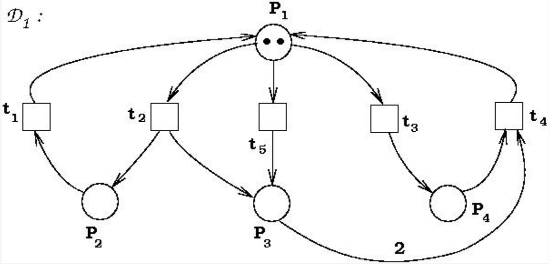
ATPN 2008, Xi'an, China

# Outline

# Timed Petri Net: An Informal Introduction

Statics: **Petri Net**



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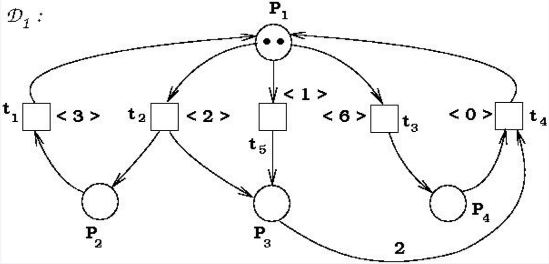

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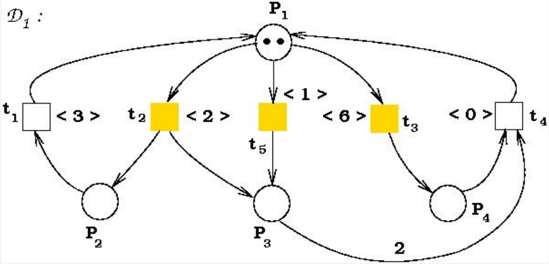


**firing mode: maximal step**

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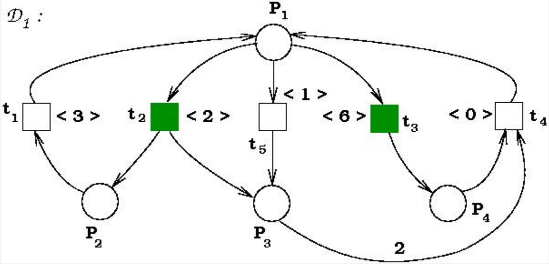


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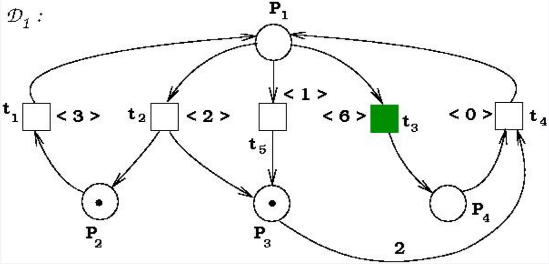


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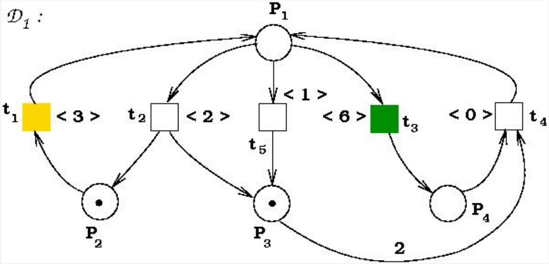


**firing mode: maximal step**

A formal definition of a Timed Petri Net can be found in the Appendix, Part II.

# Timed Petri Nets and Counter Machines

**Remark:**

The power of the Timed Petri Nets is equal to the power of the Turing Machines.

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The power of the Timed Petri Nets is equal to the power of the Turing Machines.

**Idea:**

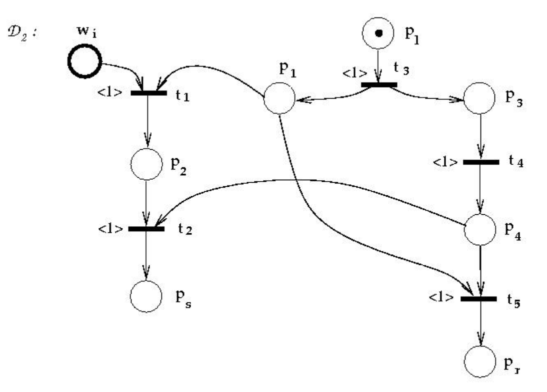
Simulation of an arbitrary Counter Machine with a Timed Petri Net.

**Sufficiently:** To simulate the command

l:*DEC(i)*:r:s (**zero-test**).

# Zero-test

Zero-test ( l:*DEC(i)*:r:s ) for Timed PN with **firing mode maximal step**



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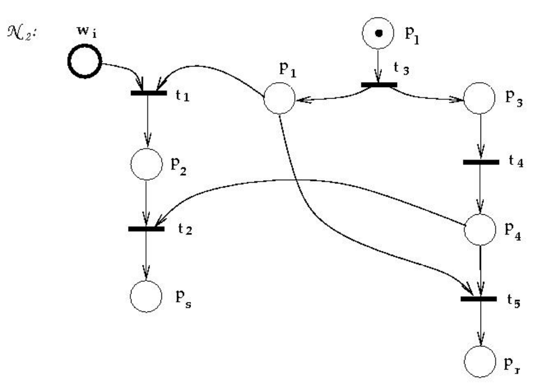
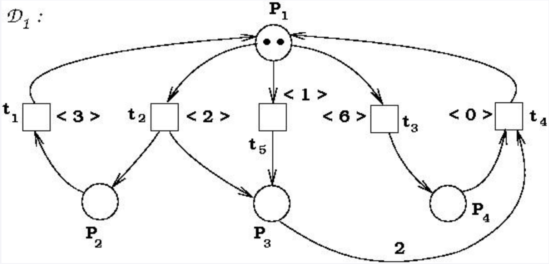


# Reachability graph



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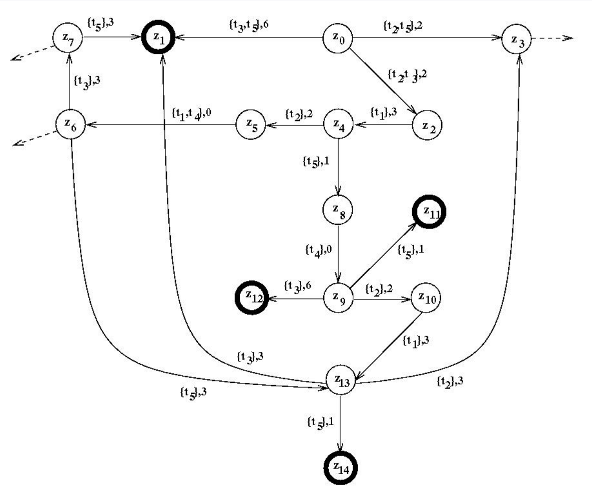


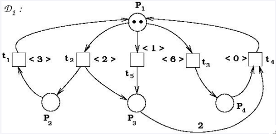

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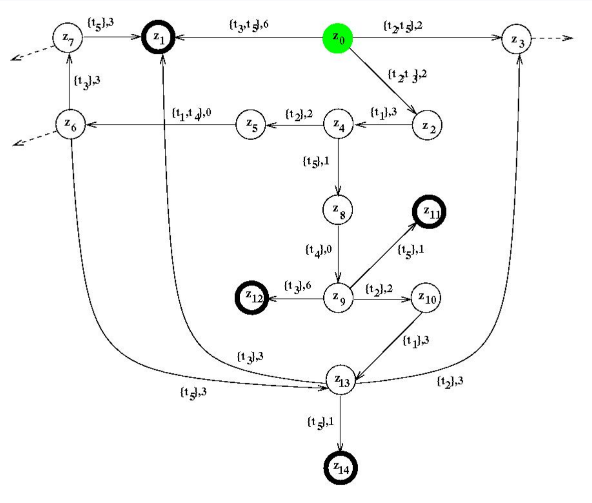


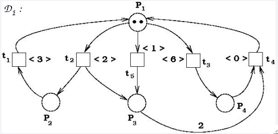

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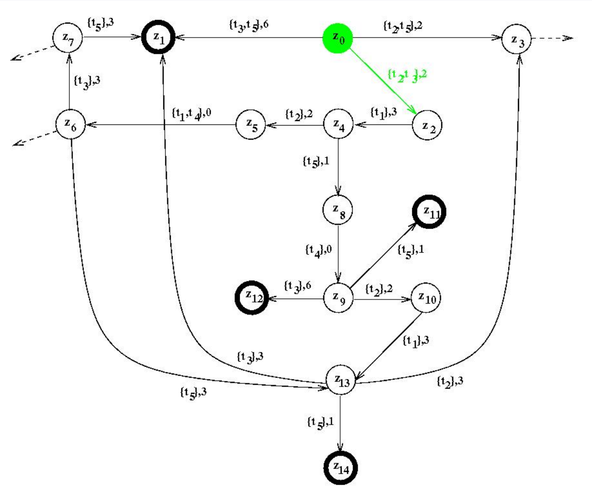


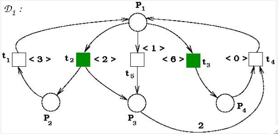

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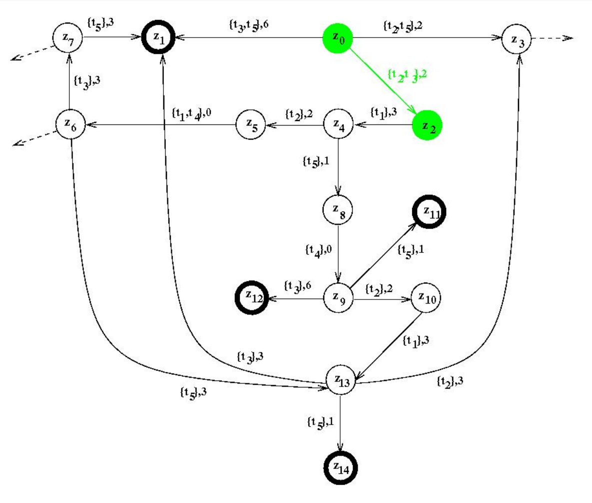


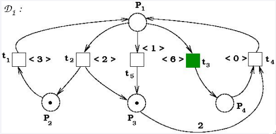

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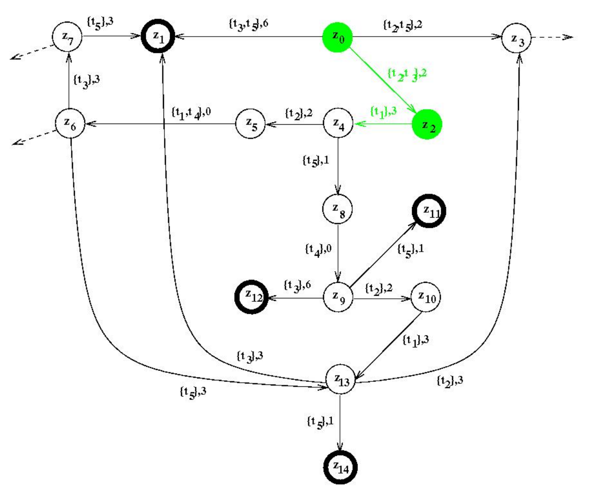


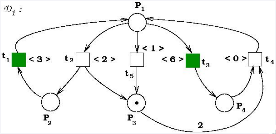

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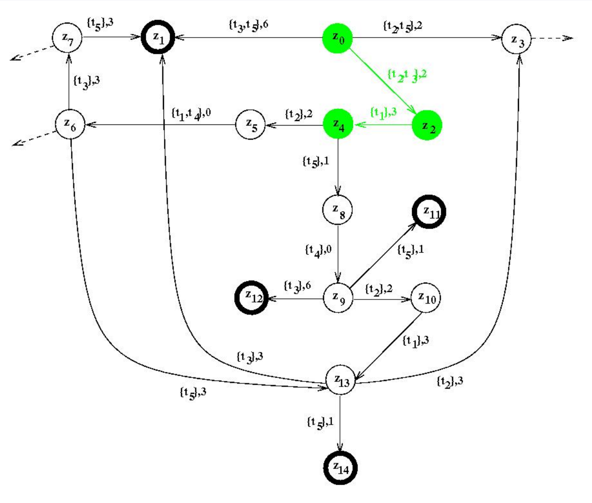


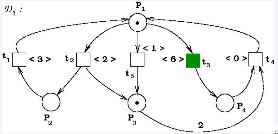

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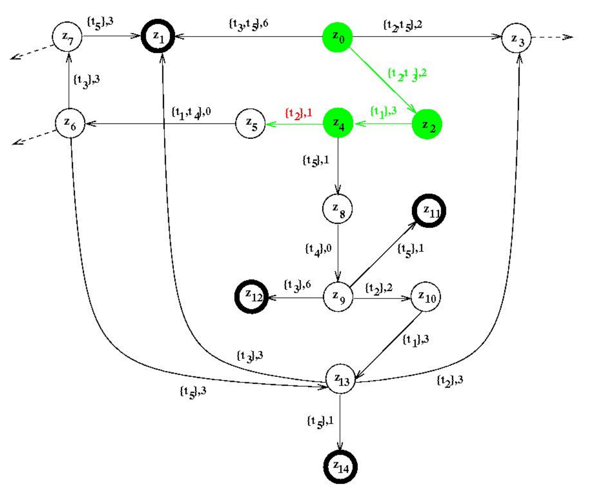


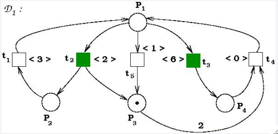

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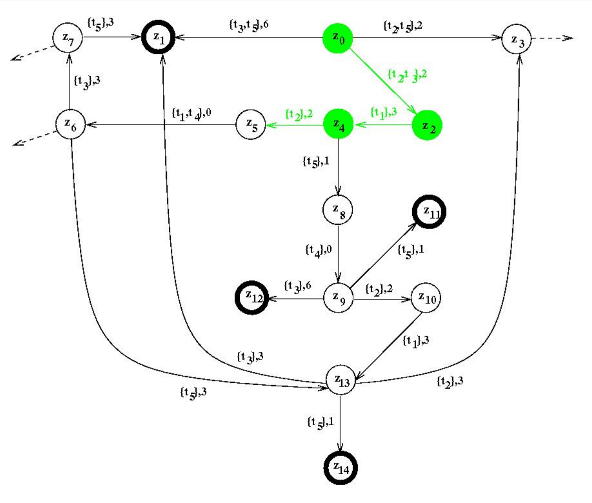


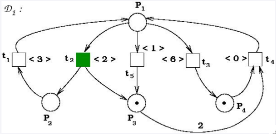

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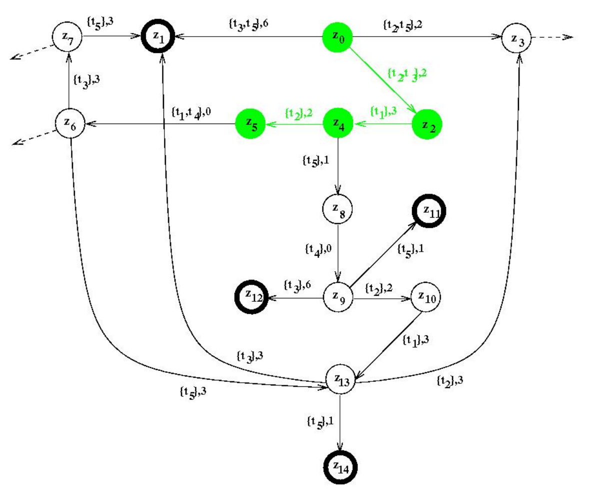


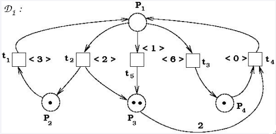

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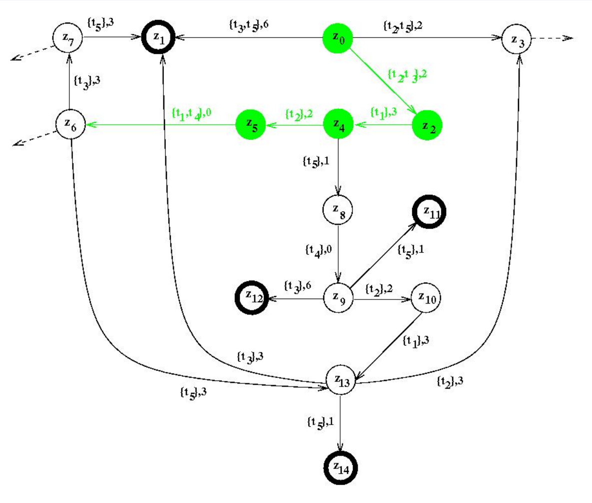


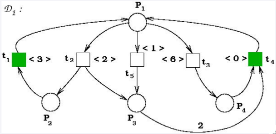

# State Equation in classical PN

Let $\mathcal{N}$ be a classical PN with

- $m_1$ and $m_2$ two markings in $\mathcal{N}$,
- $\sigma = t_1 \ldots t_n$ a firing sequence, and
- $m_1 \xrightarrow{\sigma} m_2$.

Then it holds:

$$m_2 = m_1 + C \cdot \pi_\sigma, \text{ (state equation)}$$

where $C$ is the incidence matrix of $\mathcal{N}$ and $\pi_\sigma$ is the Parikh vector of $\sigma$.

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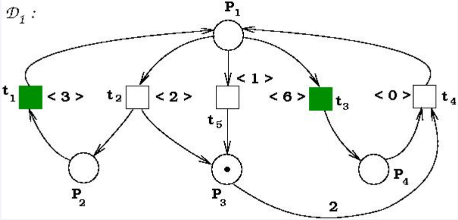
In each PN $\mathcal{N}$ with initial marking $m_0$ it holds:
If $m \neq m_0 + C \cdot \pi_\sigma$ then $m$ is not reachable in $\mathcal{N}$.

# Extended Form of a Place Marking



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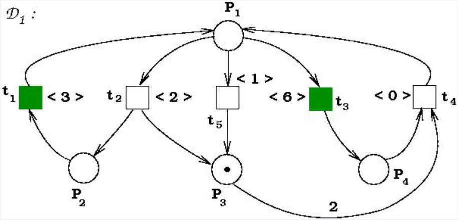
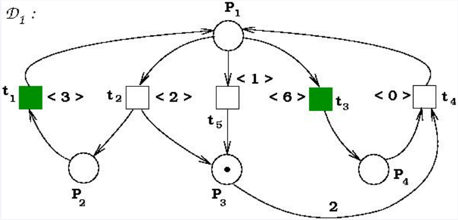



$$m = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

extended form
of the *p*-markings *m*

# Extended Form of a Place Marking




$$m = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{matrix}$$

extended form of the $p$-markings $m$

after

0 1 2 3 4 5 6

time units

# Time Dependent State Equation

**Theorem**

Let $\mathcal{D}$ be a Timed Petri Net, $z^{(0)}$ be the initial state in extended form and

$$z^{(0)} \xrightarrow{\mathfrak{G}_1} \hat{z}^{(1)} \xrightarrow[1]{} \tilde{z}^{(1)} \xrightarrow{\mathfrak{G}_2} \hat{z}^{(2)} \xrightarrow[1]{} \ldots \xrightarrow{\mathfrak{G}_n} z^{(n)}$$

be a firing sequence ($\mathfrak{G}_i$ is a multiset for each $i$). Then, it holds:

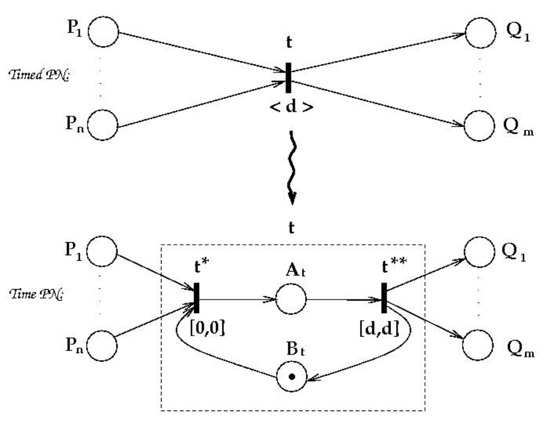
$$m^{(n)} = m^{(0)} \cdot R^{n-1} + C \cdot \Psi_\sigma. \qquad \text{State equation}$$

$$z^{(0)} \xrightarrow{\mathfrak{G}_1} \hat{z}^{(1)} \underset{1}{\longrightarrow} \tilde{z}^{(1)} \xrightarrow{\mathfrak{G}_2} \hat{z}^{(2)} \underset{1}{\longrightarrow} \dots \xrightarrow{\mathfrak{G}_n} z^{(n)}$$

$$m^{(n)} = m^{(0)} \cdot R^{n-1} + C \cdot \Psi_\sigma.$$ State equation

- $m^{(n)}$ and $m^{(0)}$ are place markings in extended form
- $R$ is the progress matrix for $\mathcal{D}$.
- $C$ is the incidence matrix of $\mathcal{D}$ in extended form
- $\Psi_\sigma$ is the Parikh matix of the sequence $\sigma = \mathfrak{G}_1 \; \mathfrak{G}_2 \; \dots \mathfrak{G}_n$ of multisets of transitions.

# Transformation Timed PN –> Time PN



# Transformation Timed PN –> Time PN

If in the Timed PN a firing duration is zero, then some problems are possible:

# Transformation Timed PN –> Time PN

**If in the Timed PN a firing duration is zero, then some problems are possible:**

It is possible that both

- the set of the reachable *p*- markings and
- the set of firing sequences

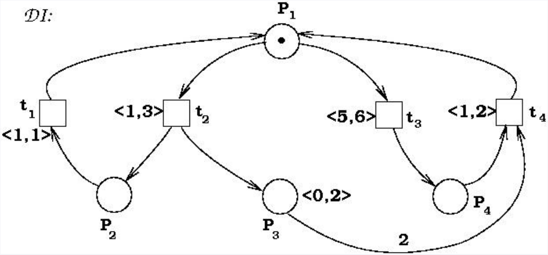
in the derived TPN are supersets of the corresponding sets in the Timed PN.

# Sufficient Conditions for the Nonreachability of *p*-markings

The *p*-marking *m*

- does not satisfy a state equation.
- does not satisfy the maximality condition for the firing rule.

# Duration Interval Petri Nets



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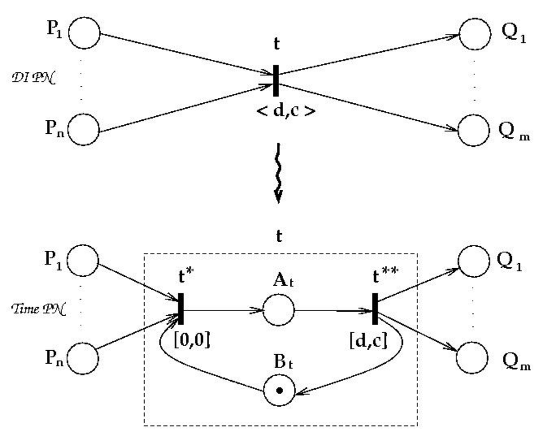
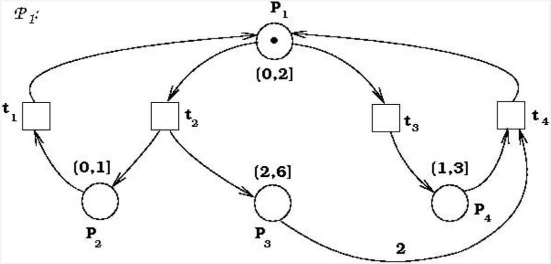
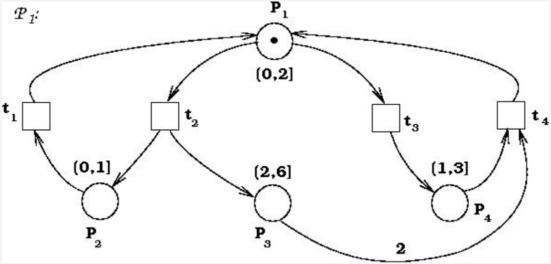


# Petri Nets with Time Dependent Places



# Petri Nets with Time Dependent Places




This class of time dependent Petri Nets is equivalent to the classical Petri Nets (and therefore not equivalent to Turing Machines).

**Theorem:** Let $\mathcal{P}$ be a PN with time dependent places and $T$ be the set of its transitions. Let

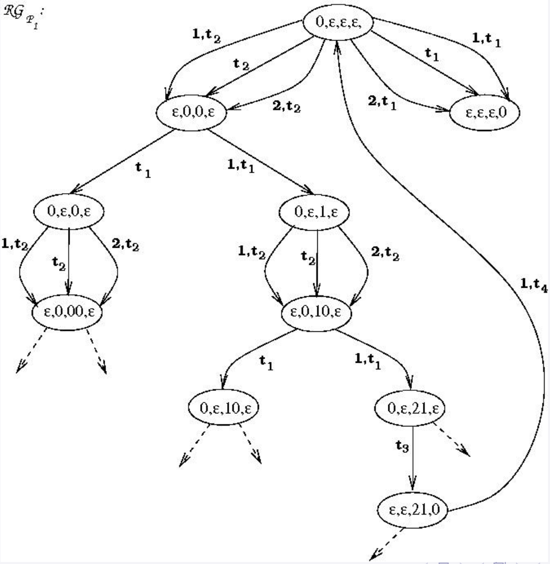
$$\sigma(\tau) = \tau_0 t_1 \tau_1 t_2 \tau_2 \ldots \tau_{n-1} t_n$$

be a feasable run in $\mathcal{P}$ with $\tau_i \in \mathbb{R}_0^+$, for all $i$, $0 \leq i \leq n-1$. Than there exists a feasable run

$$\sigma(\tau^*) = \tau_0^* t_1 \tau_1^* t_2 \tau_2^* \ldots \tau_{n-1}^* t_n$$

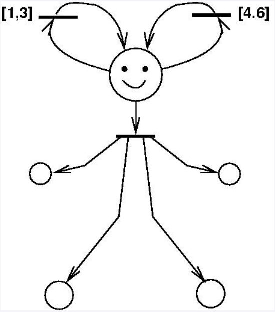
in $\mathcal{P}$ and $\tau_i^* \in \mathbb{N}$, for all $i$, $0 \leq i \leq n-1$.

# Reachability Graph (Segment)



- Given: Time dependent Petri Net
- Aim: Analysis of the time dependent Petri Net
- Problem: Infinite (dense) state space, TM-Equivalence
- Solution:
  - Parametrisation and discretisation of the state space.
  - Definition of an reachability graph.
  - Structurally restricted classes of time dependent Petri Nets.
  - Time dependent state equation.

**Thank you!**



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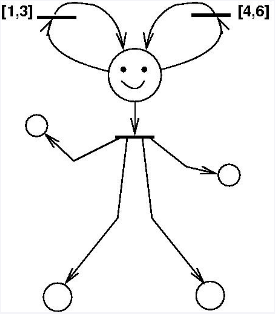

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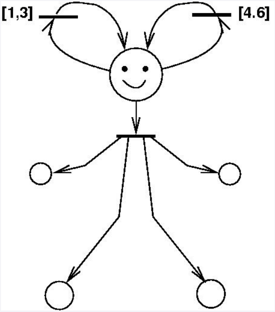


# Timed Petri Net – Statics

## Definition (Timed Petri Net)

The 6-tupel $\mathcal{D} = (P, T, F, V, m_0, D)$ is called Timed Petri Net (short: DPN), iff

- the 5- $= (P, T, F, V, m_0) =: S(\mathcal{D})$ is a Petri Net
- $D : T \longrightarrow \mathbb{Q}_0^+$, called duration function.

# Timed Petri Net – Dynamics

## Definition (state)

A pair $z = (m, u)$ is called a state in the DPN $\mathcal{D}$ iff

- $m$ is a marking in $S(\mathcal{D})$ and
- $u : T \longrightarrow \mathbb{R}_0^+$ with
  $\forall t\ (\ (t \in T \wedge t^- \leq m) \ \longrightarrow\ u(t) \leq D(t)\ ).$

# Timed Petri Net – Dynamics

## Definition (maximal step)

Let $z = (m, u)$ be a state in the DPN $\mathcal{D}$ and let $T$ be the set of its transitions. Then, the set $M$ is called a maximal set in $z$ iff

1. $M \subseteq T$,
2. $\forall t\,(\,t \in M \longrightarrow u(t) = 0\,)$,
3. $\sum\limits_{t \in M} t^- \leq m$,
4. $\forall \hat{t}(\,(\,\hat{t} \in T \wedge \hat{t} \notin M \wedge \hat{t}^- \leq m \wedge u(\hat{t}\,) = 0\,) \longrightarrow (\sum\limits_{t \in M} t^- + \hat{t}^-) \not\leq m)$.

# Timed Petri Net – Dynamics

## Definition (firing)

Let $z_1 = (m_1, u_1)$ be a state in the DPN $\mathcal{D}$ and let $M \subseteq T$ holds. Than $M$ can fire in $z_1$ (denoted by: $z_1 \xrightarrow{M}$ ) iff $M$ a maximal step in $z_1$.
After firing of $M$ the DPN $\mathcal{D}$ is in the state $z_2 = (m_2, u_2)$ (denoted by: $z_1 \xrightarrow{M} z_2$ ) with:

(1) $m_2 := m_1 - \sum\limits_{t \in M} t^- + \sum\limits_{\substack{t \in M, \\ D(t)=0}} t^+,$

(2) $u_2(t) := \begin{cases} D(t) & \text{, if } t \in M \\ u_1(t) & \text{, else} \end{cases}.$

# Timed Petri Net – Dynamics

## Definition (time elapsing)

Let $z_1 = (m_1, u_1)$ be a state in the DPN $\mathcal{D}$. Then one time unit can elapse in $\mathcal{D}$ (denoted by : $z_1 \xrightarrow{1}$) iff

$$\forall t \left( (t \in T \wedge u_1(t) = 0) \longrightarrow t^- \not\leq m_1 \right).$$

After the elapsing of one time unit the DPN $\mathcal{D}$ is in the state $z_2 = (m_2, u_2)$ (denoted by: $z_1 \xrightarrow{1} z_2$ ) with:

1. $m_2 := m_1 + \sum\limits_{\substack{t \in T, \\ u_1(t)=1}} t^+,$
2. $u_2(t) := \begin{cases} u_1(t) - 1 & \text{, if } u_1(t) \geq 1 \\ 0 & \text{, else} \end{cases}.$

**Definition (incidence matrix)**

Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri Net. The matrix

$$C_{\mathcal{N}} := \big( - t_j^-(p_i) + t_j^+(p_i) \big), \quad i = 1 \ldots |P|,\ j = 1 \ldots |T|$$

is called the **incidence matrix** of $\mathcal{N}$.

### Definition (Parikh vektor)

Let $\mathcal{N} = (P, T, F, V, m_0)$ be a PN and $\sigma = t_1 \ldots t_n$ be a firing sequence in $\mathcal{N}$. The vector $\pi \in \mathbb{N}^{|T|}$ with

$$\pi(t) := \text{ number of appearances of the transition } t \text{ in the sequence } \sigma$$

is called the **Parikh vector** of $\sigma$.

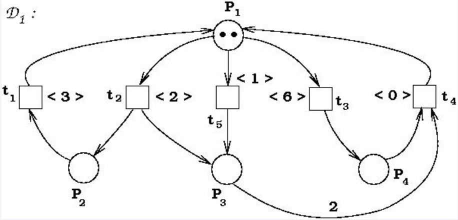
# The Progress Matrix for $\mathcal{D}_1$

$$R_{\mathcal{D}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$R_{\mathcal{D}_1}$ is a $d \times d$ matrix and $d = 7 = \underbrace{\text{max. duration in } R_{\mathcal{D}_1}}_{6} + 1$

# The Incidence Matrix of $\mathcal{D}_1$ in Extended Form

$$C_{\mathcal{D}_1} = \begin{pmatrix} 0\,0\,0\,1\,0\,0\,0 & -1\,0\,0\,0\,0\,0\,0 & -1\,0\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0\,0 & -1\,0\,0\,0\,0\,0\,0 \\ -1\,0\,0\,0\,0\,0\,0 & 0\,0\,1\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,1\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 & -2\,0\,0\,0\,0\,0\,0 & 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,1 & -1\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 \end{pmatrix}$$



# The Bag-Matrix of a (global) step in a Timed PN

The matrix $G^{(i)}$ is the **bag-matrix** of the (global) firing step $\mathfrak{G}_i$ iff

$$G^{(i)} = \begin{pmatrix} G_{(1)} \\ G_{(2)} \\ \vdots \\ G_{(|T|)} \end{pmatrix}, \quad G_{(s)} = \kappa_s^{(i)} \cdot E_d, \text{ where}$$

$\kappa_s^{(i)}$ is the number of appearance of $t_s$ in $\mathfrak{G}_i$ and $E_d$ is the unit matrix of the dimension $d$.

# The Bag-Matrix of a (global) step in a Timed PN

## The Bag-Matrix of the (global) step $\mathfrak{G}_1 = \{t_2, t_3\}$ in $\mathcal{D}_1$

In $\mathcal{D}_1$ is $d = 7$ and $|T| = 5$.

In $\mathfrak{G}_1$ is $\kappa_1^{(1)} = \kappa_4^{(1)} = \kappa_5^{(1)} = 0$ and $\kappa_2^{(1)} = \kappa_3^{(1)} = 1$.

Finally,

$$G^{(1)} = \begin{pmatrix} 0 \cdot E_7 \\ 1 \cdot E_7 \\ 1 \cdot E_7 \\ 0 \cdot E_7 \\ 0 \cdot E_7 \end{pmatrix}$$

# The Parikh Matrix of a $\sigma$ in a Timed PN

$\Psi$ is the **Parikh Matrix** of the sequence $\sigma = \mathfrak{G}_1\ \mathfrak{G}_2\ \ldots \mathfrak{G}_n$ of (global) steps, i.e.

$$z^{(0)} \xrightarrow{\mathfrak{G}_1} \hat{z}^{(1)} \xrightarrow[1]{} \tilde{z}^{(1)} \xrightarrow{\mathfrak{G}_2} \hat{z}^{(2)} \xrightarrow[1]{} \ldots \xrightarrow{\mathfrak{G}_n} \hat{z}^{(n)},$$

iff:

$$\Psi_\sigma := \sum_{i=1}^{n} G^{(i)} \cdot R^{n-i}.$$