On Dynamics in Selfish Network Creation

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SPAA’13, Montréal
Models of Selfish Network Creation (1)

- \( n \) selfish agents want to create a \textit{connected} undirected network \( G = (V, E) \)
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- $n$ selfish agents want to create a *connected* undirected network $G = (V, E)$
- agents want to minimize cost for network usage while maximizing connection quality
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cost of an agent $u$ in network $(G, \alpha)$:

$\text{cost}(u) = \text{edgecost}(u) + \text{distancecost}(u)$

$= \alpha \cdot (\#\text{edges bought by agent } u) + \text{distancecost}(u)$
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**SUM-Version:** [Fabrikant et al., PODC'03]

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\text{distancecost}(u) = \begin{cases} 
  \sum_{v \in V(G)} d_G(u, v), & \text{if } (G, \alpha) \text{ is connected} \\
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• pure strategy $S_u$ of agent $u$: $S_u \subseteq V \setminus \{u\}$. Depending on the model we have:
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- $S$ is vector of strategies of all agents
  - $S$ and parameter $\alpha$ determines network $(G, \alpha)$
  - network $(G, \alpha)$ with edge ownerships determines $S$
Models of Selfish Network Creation (2)
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Swap Game (SG)  
[Alon et al. SPAA’10]

- no edge-owners
- no edge-cost
- only single edge-swaps
- both endpoints can swap

⇒ Swap Equilibrium
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- each edge costs $\alpha$
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**Open Problem:**

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**Open Problem:**
How can agents **find** equilibrium networks?

⇒ we focus on the network creation *process*
  • we analyze the most natural approach:

**Distributed Local Search:**
  • start with any connected network
  • at every step one agent is allowed to move (agent chosen at random or random max cost agent)
  • moving agent performs move to best response strategy
  • iterate until no agent wants to change strategy
Classifying Games According to their Dynamics

• guaranteed convergence:
  • FIPG: games having the finite improvement property (FIP)
    • FIP \iff generalized ordinal potential function exists
    • all sequences of improving moves are finite
    • no better response cycle
  • poly-FIPG: FIP + convergence in polynomially many rounds

• possible convergence:
  • WAG: games which are weakly acyclic
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- FIPG
- poly-FIPG
- potential games
- BR-WAG
- WAG

- Sum-BG ∈ FIPG via better response cycle [Brandes et al. WINE’08]
- How fast converge bounded-budget versions? [Ehsani et al. SPAA’11]
- bounded-budget version is ASG (agents use up their budgets)
- Sum-SG on trees ∈ poly-FIPG, / ∈ FIPG otherwise [L. SAGT’11]
- on trees: any improving sequence has length $O(n^3)$, speed-up to $O(n)$ if max cost agents play best response

- Max-BG ∈ FIPG via better response cycle [Bilò et al. WINE’12]
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- **MAX-BG ∉ FIPG** via better response cycle [Bilò et al. WINE’12]
  ⇒ for most variants nothing known for **best** response dynamics
Our Results

**Max-Swap Game**

- on trees: poly-FIPG, at most $O(n^3)$ steps,
- in general: $/ \in FIPG$ via best response cycle
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Asymmetric SG
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**Greedy Buy Game**

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Greedy Buy Game
- $\text{Sum}$: best response cycle

Buy Game
- $\text{Sum}$: best response cycle
Our Results

Max-Swap Game
- on trees: poly-FIPG, at most $O(n^3)$ steps, speed-up to $O(n \log n)$
- in general: $\notin$ FIPG via best response cycle

Asymmetric SG
- SG-results on trees carry over for Sum and Max
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\begin{center}
\begin{tikzpicture}
\node at (0,0) [shape=circle,draw] (u) {$u$};
\node at (1,0) [shape=circle,draw] (v) {$v$};
\node at (2,0) [shape=circle,draw] (w) {$w$};
\node at (0,-1) [shape=circle,draw] (A) {$A$};
\node at (2,-1) [shape=circle,draw] (B) {$B$};
\draw (u) -- (v) -- (w);
\draw (A) -- (u);
\draw (B) -- (w);
\end{tikzpicture}
\end{center}
Details for **Max Swap Game on Trees**

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![Diagram](attachment:image.png)
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**Definition: Sorted Cost Vector**

$$\vec{c}_G = (\gamma^1_G, \ldots, \gamma^n_G),$$

where $\gamma^i_G$ is cost of agent with $i$-th highest cost in network $G$.

- assume improving swap $uv \rightarrow uw$
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\[ T : \quad \begin{array}{c}
A \\
\uparrow \\
v \\
\downarrow \\
B \\
\end{array} \quad \begin{array}{c}
w \\
\end{array} \]

\[ T' : \quad \begin{array}{c}
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v \\
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B \\
\end{array} \quad \begin{array}{c}
w \\
\end{array} \]

\[ x \rightarrow_{\text{red}} w \]
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  $$\Rightarrow$$ diameter cannot increase
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- consider tree network $T$ having diameter $D \geq 4$:

**Lemma**

After $\frac{n^*D - D^2}{2}$ steps in $T$, diameter must decrease.
Details for **Max** Swap Game on Trees

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- Equilibria are stars or double-stars [Alon et al. SPAA’10]
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- Equilibria are stars or double-stars [Alon et al. SPAA’10]
  $\Rightarrow$ process must converge after $O(n^3)$ steps.
Details for Asymmetric Swap Games

Asymmetric Swap Games

- SG-results on trees carry over for Sum and Max
- in general: Sum ∉ WAG, Max ∉ FIPG
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Remember:

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**Open Problem** [Ehsani et al. SPAA'11]
Determine convergence speed of $\text{Sum}$ and $\text{Max}$ in bounded budget version.

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  $\Rightarrow$ sharp boundary between convergence and non-convergence
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Details for (Greedy) Buy Games

Greedy Buy Game
- **Sum**: best response cycle
- **Max**: best response cycle
- **Sum** and **Max** ∉ WAG on general host graphs
- extensive simulations show convergence in < 8n steps

Buy Game
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Details for (Greedy) Buy Games

We give best response cycle for $7 < \alpha < 8$:

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We give best response cycle for $7 < \alpha < 8$:

\[ \begin{array}{ccc}
  a & \rightarrow & b \\
  b & \rightarrow & c \\
  c & \rightarrow & d \\
  d & \rightarrow & e \\
  e & \rightarrow & f \\
  f & \rightarrow & g \\
  g & \rightarrow & h \\
  h & \rightarrow & i \\
  i & \rightarrow & j \\
  j & \rightarrow & k \\
  k & \rightarrow & l \\
  l & \rightarrow & m \\
  m & \rightarrow & n \\
  n & \rightarrow & o \\
  o & \rightarrow & p \\
  p & \rightarrow & q \\
  q & \rightarrow & r \\
  r & \rightarrow & s \\
  s & \rightarrow & t \\
  t & \rightarrow & u \\
  u & \rightarrow & v \\
  v & \rightarrow & w \\
  w & \rightarrow & x \\
  x & \rightarrow & y \\
  y & \rightarrow & z \\
  z & \rightarrow & a \\
\end{array} \]

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$gf \rightarrow gc$

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\[
\begin{align*}
&\text{Greedy Buy Game} \\
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\end{align*}
\]

\[
\begin{align*}
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&\text{•} \, \textbf{Sum}: \text{best response cycle} \\
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&\text{•} \, \textbf{Sum} \text{ and } \textbf{Max} \notin \text{WAG on general host graphs} \\
&\text{•} \, \text{bilateral Buy Game: } \textbf{Sum} \notin \text{WAG}, \textbf{Max} \notin \text{FIPG}
\end{align*}
\]
Details for (Greedy) Buy Games

We give best response cycle for $7 < \alpha < 8$:

\[ a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow f \rightarrow c \rightarrow b \rightarrow a \]

Greedy Buy Game

- **Sum**: best response cycle
- **Max**: best response cycle
- **Sum** and **Max** $\notin$ WAG on general host graphs
- extensive simulations show convergence in $< 8n$ steps

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![Diagram showing best response cycle for (Greedy) Buy Games with $7 < \alpha < 8$.]

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We give best response cycle for $7 < \alpha < 8$:

- $g \rightarrow f$ buys $fb$
- $f$ buys $fb$
- $c$ rem. $cb$
- $gc \rightarrow gf$
- $c$ buys $cb$

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  ⇒ despite millions of runs: no cyclic instance found
  ⇒ suprisingly fast convergence: $\text{SUM} < 7n$ moves, $\text{MAX} < 8n$

### Greedy Buy Game

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Details for (Greedy) Buy Games

Max # of steps until convergence, SUM version

- \(m=n, a=n/10, \text{max cost}\)
- \(m=n, a=n/4, \text{max cost}\)
- \(m=n, a=n, \text{max cost}\)
Details for (Greedy) Buy Games

Max # of steps until convergence, SUM version

- $m=n$, $a=n/10$, max cost
- $m=n$, $a=n/4$, max cost
- $m=n$, $a=n$, max cost
- $m=4n$, $a=n/10$, max cost
- $m=4n$, $a=n/4$, max cost
- $m=4n$, $a=n$, max cost
Details for (Greedy) Buy Games

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- $m=n$, $a=n$, max cost
- $m=4n$, $a=n/10$, max cost
- $m=4n$, $a=n/4$, max cost
- $m=4n$, $a=n$, max cost
- $m=n$, $a=n/10$, random
- $m=n$, $a=n/4$, random
- $m=n$, $a=n$, random
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- $m=n$, $a=n$, max cost
- $m=4n$, $a=n/10$, max cost
- $m=4n$, $a=n/4$, max cost
- $m=4n$, $a=n$, max cost
- $m=n$, $a=n/10$, random
- $m=n$, $a=n/4$, random
- $m=n$, $a=n$, random
- $m=4n$, $a=n/10$, random
- $m=4n$, $a=n/4$, random
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- $m=4n, a=n/4$, max cost
- $m=4n, a=n$, max cost
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- $m=4n, a=n/10$, random
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- $m=4n, a=n$, random

$f(n) = 7n$
Details for (Greedy) Buy Games

Max # of steps until convergence, MAX version

- $m=n$, $a=n/10$, max cost
- $m=n$, $a=n/4$, max cost
- $m=n$, $a=n$, max cost
Details for (Greedy) Buy Games

Max # of steps until convergence, MAX version

- \( m=n, a=n/10, \) max cost
- \( m=n, a=n/4, \) max cost
- \( m=n, a=n, \) max cost
- \( m=4n, a=n/10, \) max cost
- \( m=4n, a=n/4, \) max cost
- \( m=4n, a=n, \) max cost

Graph showing the relationship between steps and agents for different configurations.

Agents vs. Steps graph with markers for each condition.
Details for (Greedy) Buy Games

Max # of steps until convergence, MAX version

- \(m=n, a=n/10\), max cost
- \(m=n, a=n/4\), max cost
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$f(n) = 8n$
Our Results

Max-Swap Game
- on trees: poly-FIPG, at most $O(n^3)$ steps, speed-up to $O(n \log n)$
- in general: $\not\in$ FIPG via best response cycle

Asymmetric SG
- SG-results on trees carry over for $\text{Sum}$ and $\text{Max}$
- in general: $\text{Sum} \not\in \text{WAG}$, $\text{Max} \not\in \text{FIPG}$
- solve open problem [SPAA'11]
- promising empirical results

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Open Problems

- The Sum-(G)BG and Max-(G)BG are not weakly acyclic.
- Give best response cycle where in every step exactly one agent is unhappy and this agent has exactly one improving move.
- Open Problem: Why do dynamics in (Greedy) Buy Games converge so fast?
- Open Problem: Is convergence to approximate equilibrium guaranteed? If so, for which approx-factor?
Open Problems

Conjecture

The $\text{SUM-}(G)BG$ and $\text{MAX-}(G)BG$ are not weakly acyclic.
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