Directed Tree-Width Examples

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Abstract

In [1] Johnson, Robertson, Seymour and Thomas define the notion of directed tree-width $\mathrm{dtw}(D)$ of a directed graph D. They ask whether $\mathrm{dtw}(D) \geq k-1$ implies that D has a haven of order k. A negative answer is given. Furthermore they define a generalisation of the robber and cops game of [3] to digraphs. They ask whether it is true that if k cops can catch the robber on a digraph, then they can do so robber-monotonely. Again a negative answer is given. We also show that contraction of butterfly edges can increase directed tree-width.

1 Definitions

All graphs and digraphs are finite. We recall the most important definitions from [1][1]. Let D be a digraph. A set $S \subseteq D \setminus Z$ is *Z-normal*, if there is no directed walk in $D \setminus Z$ with first and last vertex in S that uses a vertex from $D \setminus (Z \cup S)$. Consider the digraph D depicted in Figure 1, where $V(D) = \{0,1,2,3\}$ and $E(D) = \{(0,1),(1,0),(0,2),(2,0),(0,3),(3,0),(1,3),(3,1)\}$. Then $\{1,2\}$ is $\{0\}$-normal, but $\{2,3\}$ is not $\{0\}$-normal.

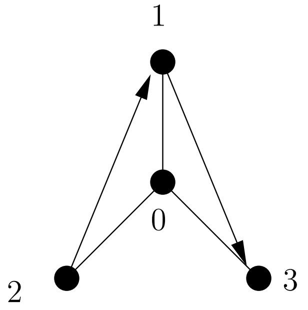

Figure 1: A digraph. $\{1,2\}$ is $\{0\}$-normal, $\{2,3\}$ is not $\{0\}$-normal.

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[1] Note that there is an addendum [2] to [1]. The addendum concerns the algorithmic aspects of diredted tree-width which are not investigateed in this paper.

An *arborescence* T is a directed, rooted tree, where all edges are directed away from the root. We abuse notation by identifying T and $V(T)$.

An *arboreal decomposition* of a digraph D is a triple (T, W, X), where T is an arborescence, and $W = (W_t)_{t \in T}$ and $X = (X_e)_{e \in E(T)}$ are families of subsets of $V(D)$ such that:

(1) W is a partition of $V(D)$ into nonempty sets.

(2) For each edge $e \in E(T)$ the set $\bigcup\{W_t \mid t > e\}$ is X_e-normal.

Here $t > e$ means that there is a directed walk (possibly of length 0) from the head of e to t. The *width* of a node $t \in T$ is

$$\mathrm{w}(t) := \left| W_t \cup \bigcup_{e \ni t} X_e \right|$$

The *width* of an arboreal decomposition of D is

$$\mathrm{w}(T, W, X) := \max\big\{\, \mathrm{w}(t) - 1 \;\big|\; t \in T \big\}.$$

The *directed tree-width of* D is

$$\mathrm{dtw}(D) := \min\big\{\, \mathrm{w}(T, B) \;\big|\; (T, B) \text{ an arboreal decomposition of } D \big\}.$$

It is easy to see that all acyclic digraphs have directed tree-width zero. An undirected graph G can be regarded as a digraph D_G by replacing each edge of G by two arrows pointing in opposite directions. G satisfies tree-width$(G) =$ dtw(D_G).[2] Let D be a digraph and let G_D be obtained from D by forgetting the direction of the edges. Then dtw$(D) \leq$ tree-width(G_D).

2 Robber-Monotonicity in Digraphs

The robber and cops game on a digraph D is a two player game with a parameter k. Player I plays k cops and player II plays the robber. Some of the cops move to at most k vertices. The robber stands on a vertex r not occupied by the cops. Then some of the cops fly in helicopters to at most k new vertices. During the flight, the robber sees which position the cops are approaching, and before they land she runs quickly along a directed cop-free path in D to a vertex r', but she may only move to r' if there is a directed cop-free path back from r' to r (so she always moves in strongly connected subsets of $V(D)$, the so-called *escape spaces*).

The cops win if at some point the robber cannot move. Otherwise the robber wins. The following is proved in Section 2 of [1].

Fact 1 *Let D be a digraph and $k \geq 0$ an integer.*
If dtw$(D) < k$ *then k cops have a winning strategy on D.*

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[2] For the definition of tree-width see [3].

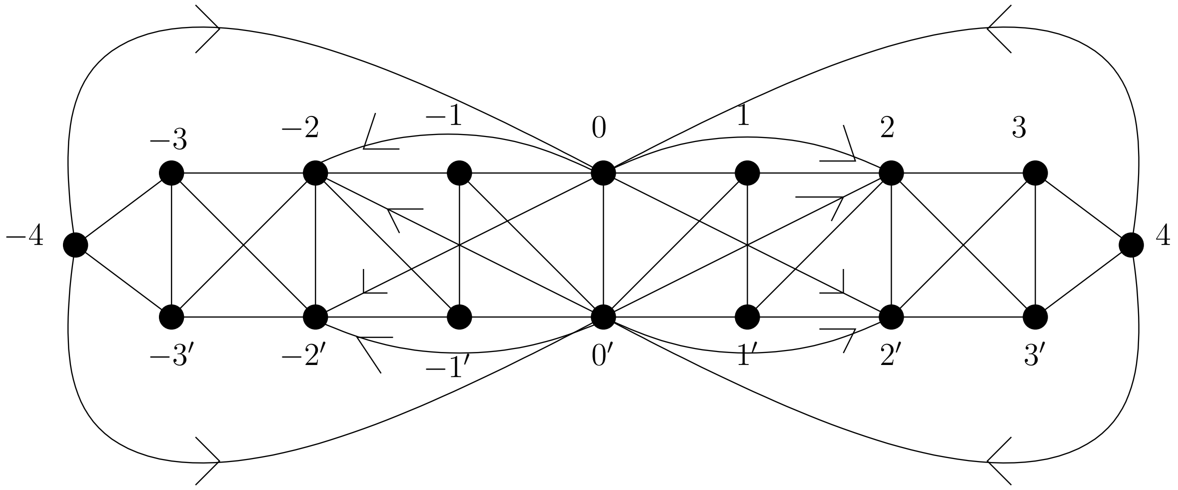

Figure 2: The digraph D^1 from Theorem 8, where four cops have a winning strategy, but five cops are needed for a robber-monotone winning strategy.

A winning strategy for the cops is *robber-monotone*, if for every sequence of cop moves $Z_1, Z_2, \ldots$ and all possible responses of the robber, the strong components of $D \backslash Z_i$ containing the robber form a nonincreasing sequence. In [1] the authors ask whether it is true that if k cops can catch the robber on a digraph, then they can do so robber-monotonely.
Figure 2 shows a digraph D^1 where four cops have a winning strategy, but they do not have a robber-monotone winning strategy.[3] The idea is, that if the cops are in position $\{2, 2', 3, 4\}$ and the robber is on $3'$, then the cops cannot catch the robber robber-monotonely, but they can catch her making a non-robber-monotone move.

Here is a winning strategy for four cops: The first position is $\{0, 0', 1, -1\}$. Due to symmetry, we may assume that the robber is on the right side. Then the cops move to $\{0, 0', 1, 4\}$, then to $\{0', 1, 1', 4\}$, $\{1, 1', 2, 4\}$, $\{1', 2, 2', 4\}$, $\{2, 2', 3, 4\}$, $\{2, 2', 3, 3'\}$ (the non-robber-monotone move), $\{3, 3', 4\}$.

We now show by a series of claims that the robber can win against four 'robber-monotone' cops.

Claim 2 *Let* $\Delta := \{0, 0', 1, -1\}$.
Then the robber can make sure that her escape space intersects Δ *until*
1) the cops occupy $\{0, 0', 1\}$*, and the robber is somewhere in* $\{1', 2, 2', 3, 3', 4\}$*, or*
2) the cops occupy $\{0, 0', -1\}$*, and the robber is in* $\{-1', -2, -2', -3, -3', -4\}$.

Proof. The robber stays on Δ as long as possible. Let Y be the first position of the four cops, in which the robber is expelled from Δ. We show that Y, R are as desired, where R is the robber's escape space with respect to Y. Let X be

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[3] The digraphs D^1 (Figure 2), D^2 (Figure 4) and D^3 (Figure 6) are modifications of the example given in [1], Section 2.

the position occupied by the cops before Y. If $\{0, 0'\} \subseteq X \cap Y$, then the robber stood on 1 w.r.t. X (or, symmetrically, on -1). Hence, $\{0, 0', 1\} \subseteq Y$ and $R \subseteq \{1', 2, 2', 3, 3', 4\}$, or $\{0, 0', -1\} \subseteq Y$ and $R \subseteq \{-1', -2, -2', -3, -3', -4\}$. Otherwise $\{0, 0'\} \nsubseteq X \cap Y$. Note that 0 and $0'$ are both connected to the two other elements of Δ. Thus, during the flight of the cops the robber can reach every element of Δ. Hence $Y = \Delta$, and the robber can choose between $R \subseteq \{1', 2, 2', 3, 3', 4\}$ and $R \subseteq \{-1', -2, -2', -3, -3', -4\}$. □

Claim 3 *Let Y be the position of the cops, $\{0, 0', 1\} \subseteq Y$, and suppose the robber's escape space R satisfies $R \subseteq \{1', 2, 2', 3, 3', 4\}$. Then the cops have to occupy $\{0, 0', 1\}$ until they move to $\{0, 0', 1, 4\}$.*

Proof. As long as the four (robber-monotone) cops do not occupy 4, each of the vertices from $\{0, 0', 1\}$ is a neighbour of the robber's escape space and cannot be released. □

Claim 4 *Let Y be the position of the cops, $\{0', 1, 4\} \subseteq Y$ and $R \subseteq \{1', 2, 2', 3, 3'\}$. Then the cops have to occupy $\{0', 1, 4\}$ until they move to $\{0', 1, 1', 4\}$.*

Claim 5 *Let Y be the position of the cops, $\{1, 1', 4\} \subseteq Y$ and $R \subseteq \{2, 2', 3, 3'\}$. Then the cops have to occupy $\{1, 1', 4\}$ until they move to $\{1, 1', 2, 4\}$.*

Claim 6 *Let Y be the position of the cops, $\{1', 2, 4\} \subseteq Y$ and $R \subseteq \{2', 3, 3'\}$. Then the cops have to occupy $\{1', 2, 4\}$ until they move to $\{1', 2, 2', 4\}$.*

Claim 7 *Let Y be the position of the cops, $\{2, 2', 4\} \subseteq Y$ and $R \subseteq \{1, 1'\}$. Then the cops have to occupy $\{2, 2', 4\}$ until they catch the robber.*

Claims 4, 5, 6 and 7 are proved just like Claim 3.

Now the fourth cop cannot catch the robber on $\{1, 1'\}$. Hence the robber has won. Altogether we have proved:

Theorem 8 *There is a digraph D^1 where 4 cops have a winning strategy but they have no robber-monotone winning strategy.*

□

3 Havens in Digraphs

Let $k \geq 0$ be an integer. A *haven* of order k in a digraph D is a function β assigning to every $Z \subseteq V(D)$ with $|Z| < k$ the vertex set of a strong component of $D \setminus Z$ in such a way that if $Z' \subseteq Z \subseteq V(D)$ with $|Z| < k$, then $\beta(Z) \subseteq \beta(Z')$.

Remark 9 *Let $k \geq 1$ be an integer. If $k - 1$ cops have a winning strategy on the digraph D, then D has no haven of order k.*

□

Hence if D is a digraph with $\mathrm{dtw}(D) < k - 1$, then by Fact 1 the digraph D has no haven of order k. In [1] the authors ask whether the converse holds. We give a counterexample for $k = 5$:

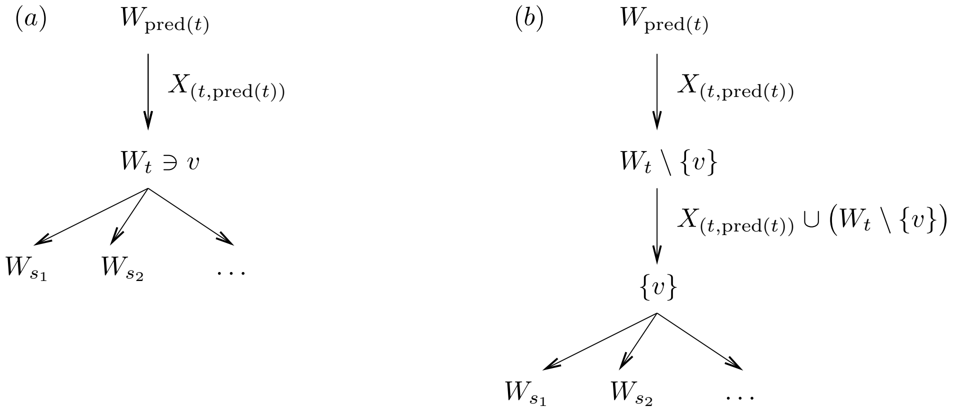

Figure 3: In the proof of Lemma 12, (a) is replaced by (b): v is removed from W_t and a new node t' with $W_{t'} = \{v\}$ is added after t.

Theorem 10 *Let* D^2 *be the digraph depicted in Figure 4. Then* $\operatorname{dtw}(D^2) \geq 4$ *and* D^2 *has no haven of order* 5.

For the proof we define an *arboreal u-decomposition* to be the variant of an arboreal decomposition that is obtained by replacing condition (2) by

(2') For each edge $e \in E(T)$ the set $\bigcup\{W_t \mid t > e\}$ is the union of strong components in $D \setminus X_e$.

We denote the corresponding width by $\operatorname{udtw}(D)$.

Lemma 11 *Let* D *be a digraph. Then* $\operatorname{udtw}(D) \leq \operatorname{dtw}(D)$.

Proof. If S is Z-normal, then S is a union of strong components of $D \setminus Z$. ☐

Lemma 12 *Let* (T, X, W) *be an arboreal u-decomposition of a digraph* D *with* $|W_t| \geq 2$ *for some* $t \in T$. *Then there exists an arboreal u-decomposition* (T', W', X') *of* D *with* $V(T') = V(T) \cup \{t'\}$ *for a node* $t' \notin T$ *and*

- $\operatorname{w}(T', W', X') \leq \operatorname{w}(T, W, X)$,
- $|W'_r| = |W_r|$ *for all* $r \in T \setminus \{t\}$,
- $|W'_t| = |W_t| - 1$, *and*
- $\left|W'_{t'}\right| = 1$.

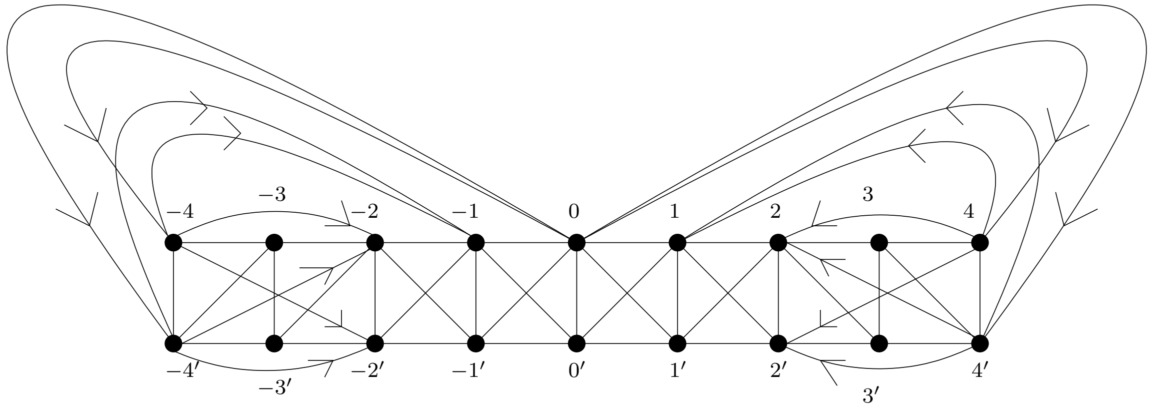

Figure 4: The digraph D^2 from Theorem 10 with $\operatorname{dtw}(D^2) \geq 4$. D^2 has no haven of order 5.

Proof. If $s \in T$ is not the root of T, by $\operatorname{pred}(s)$ we denote the unique predecessor of s in T. Let $t \in T$ satisfy $|W_t| \geq 2$. Define (T', W', X') as follows (cf. Figure 3).

$$\begin{aligned}
V(T') &:= V(T) \cup \{t'\}, \text{ for some } t' \notin T,\\
E(T') &:= \big(E(T) \setminus \{(t,s) \mid t = \operatorname{pred}(s) \text{ in } T\}\big)\\
&\qquad \cup \{(t',s) \mid t = \operatorname{pred}(s) \text{ in } T\} \cup \{(t,t')\},\\
W'_r &:= W_r \text{ for all } r \in T \setminus \{t\},\\
W'_t &:= W_t \setminus \{v\} \text{ for some } v \in W_t,\\
W'_{t'} &:= \{v\},\\
X'_e &:= X_e \text{ for all } e \in E(T) \setminus \{(t,s) \mid s \in T,\ t = \operatorname{pred}(s) \text{ in } T\},\\
X'_{(t',s)} &:= X_{(t,s)}, \text{ and}\\
X'_{(t,t')} &:= X_{(t,\operatorname{pred}(t))} \cup \big(W_t \setminus \{v\}\big).
\end{aligned}$$

It is straightforward to check that (T', W', X') satisfies (1), (2′) and all additional requirements. ☐

Corollary 13 *Let (T, X, W) be an arboreal u-decomposition of a digraph D. Then there exists an arboreal u-decomposition (T^*, W^*, X^*) of D satisfying*

- $\operatorname{w}(T^*, W^*, X^*) \leq \operatorname{w}(T, W, X)$*, and*
- $|W^*_r| = 1$ *for all* $r \in T^*$.

Proof. Repeated application of Lemma 12. ☐

Proof of Theorem 10. First we show that D^2 has no haven of order 5. By Remark 9 it suffices to show that 4 cops have a winning strategy on D^2: In the first move, the cops occupy $\{0, 0'\}$. Due to symmetry we may assume the robber is on the right hand side. The cops then move to $\{0, 0', 1, 1'\}$, $\{1, 1', 2, 2'\}$, $\{0, 1, 2, 2'\}$, $\{0, 2, 2', 3'\}$, $\{0, 2, 3, 3'\}$, $\{0, 3, 3', 4'\}$, $\{0, 3, 4, 4'\}$, catching the robber.

Now we show that $\operatorname{dtw}(D^2) \geq 4$. By Lemma 11 and Corollary 13 it suffices to show that D^2 has no arboreal u-decomposition (T, W, X) of width 3 such that all $r \in T$ satisfy $|W_r| = 1$. This is done in the following claims.
Towards a contradiction, suppose (T, W, X) is a width 3 arboreal u-decomposition of D^2 such that all $r \in T$ satisfy $|W_r| = 1$. We will identify $t \in T$ with the unique $v \in V(D^2)$ such that $W_t = \{v\}$.

Claim 14 *T has at most two leaves, namely 4 and -4.*

Proof. Let $v \in V(D^2)$ be a leaf of T and let $e \in E(T)$ be the edge with head v. Then by condition $(2')$, $\{v\}$ is a strong component in $D^2 \setminus X_e$. Since (T, W, X) is a width 3 arboreal u-decomposition it satisfies $\mathrm{w}(v) \leq 4$. It is easy to see that therefore either $v = 4$ and $X_e = \{0, 3, 4'\}$, or $v = -4$ and $X_e = \{0, -3, -4'\}$. □

Due to symmetry we may assume that 4 is a leaf of T.

Claim 15 *(T, W, X) has at most one branching node $b \in T$. If b is a branching node, then b has only two successors.*

Proof. Every branch has at least one leaf. Use Claim 14. □

Claim 16 *Suppose 4 is a leaf of T. Then (T, W, X) contains one of the configurations (a) - (d) depicted in Figure 5.*

Proof. By the proof of Claim 14, $X_{(4,\mathrm{pred}(4))} = \{0, 3, 4'\}$. If $\mathrm{pred}(4)$ branches, then we are in case (a).
Otherwise by an argument similar to the proof of Claim 14 we find $\mathrm{pred}(4) = 4'$ and $X_{(4',\mathrm{pred}(4'))} = \{0, 3, 3'\}$. If $\mathrm{pred}(4')$ branches we have case (b).
Otherwise another argument of the same type shows that $\mathrm{pred}(4') = 3$ and $X_{(3,\mathrm{pred}(3))} = \{0, 2, 3'\}$. If $\mathrm{pred}(3)$ branches we are in case (c).
Otherwise, again by a similar argument, $\mathrm{pred}(3) = 3'$ and $X_{(3',\mathrm{pred}(3'))} = \{0, 2, 2'\}$. Now we can show that $v := \mathrm{pred}(3')$ branches (so we are in case (d)): Suppose not. Then $\{v, 3, 3', 4, 4'\}$ is a union of strong components of $D^2 \setminus X_{(v,\mathrm{pred}(v))}$. Since $\{v, 3, 3', 4, 4'\}$ has at least 2 neighbours (by undirected edges) outside $\{0, 2, 2'\}$ which must be included in $X_{(v,\mathrm{pred}(v))}$, we have $\mathrm{w}(v) \geq 5$, a contradiction. □

Claim 17 *T branches and has precisely two leaves: 4 and -4.*

Proof. T must have at least one leaf, 4 or -4. By Claim 16 and symmetry there is a branching node b, and hence another leaf. □

Claim 18 *(T, W, X) contains one of the configurations of Figure 5 and one of their negative counterparts (with the same b).*

Proof. By Claim 17, Claim 16 and symmetry. □

Now it easy to check that $\mathrm{w}(b) \geq 5$, a contradiction. □

In [1] a notion of minor for digraphs is defined. The digraph D' is a *butterfly minor* of the digraph D, if D' is obtained from a subdigraph of D by contracting

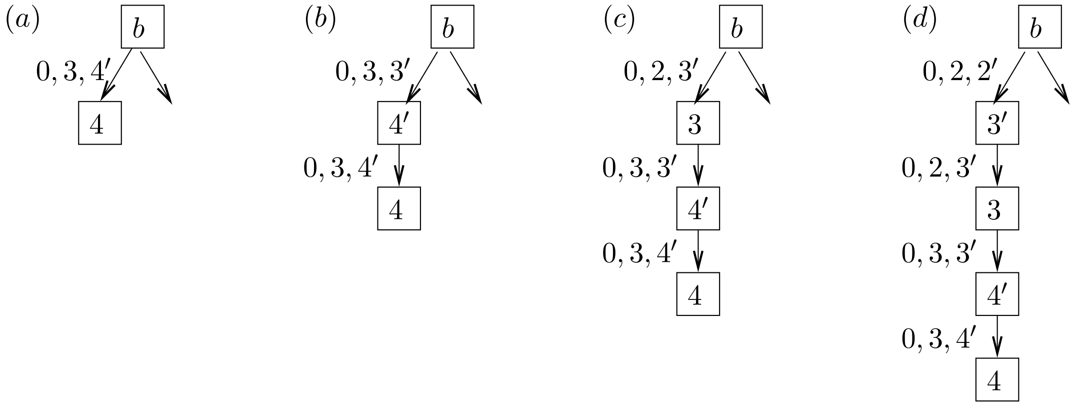

Figure 5: A width 3 tree-u-decomposition of the digraph D^2 of Figure 4 contains one these configurations.

butterfly edges, i. e. edges $e \in E(D)$ that are the only in-edge of their head or the only out-edge of their tail. The authors mention in passing that if D' is a butterfly minor of D, then $\mathrm{dtw}(D') \leq \mathrm{dtw}(D)$. The following theorem disproves this statement:

Corollary 19 *The digraph D^2 (Figure 4) is a butterfly minor of the digraph D^3 of Figure 6. Yet* $\mathrm{dtw}(D^3) \leq 3 < \mathrm{dtw}(D^2)$.

Proof. Note that D^2 can be obtained from D^3 by contracting the two butterfly edges $(0, \pi)$ and $(0, -\pi)$. The arboreal decomposition of Figure 6 shows that $\mathrm{dtw}(D^3) \leq 3$. By Theorem 10, $\mathrm{dtw}(D^2) \geq 4$. ☐

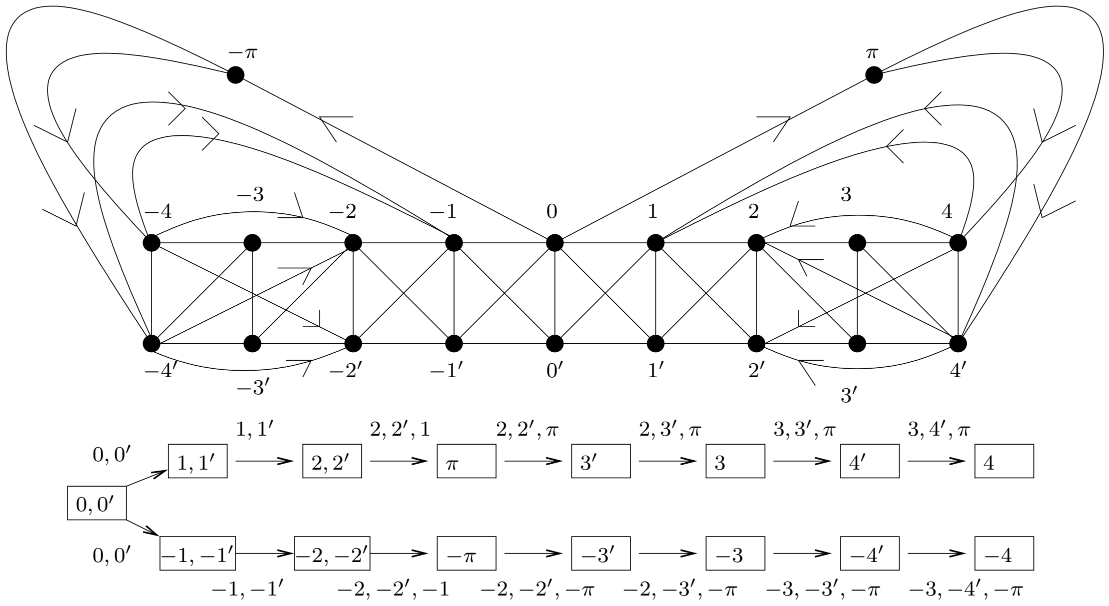

Figure 6: The digraph D^3 with $\mathrm{dtw}(D^3) \leq 3$ and an arboreal decomposition witnessing this. D^2 is a butterfly minor of D^3.

References

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